On Some Fractional Stochastic
Integral Equations

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Abstract

The Ito-Skorohod stochastic equation with fractional integral are studied. The existence and uniqueness of the fractional stochastic integral equations are established.

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1 Introduction

Let $T = [0, 1]$ be the unit interval. Consider $(W(t))_{t \in T}$ be a standard Wiener process on the canonical Wiener space $(\Omega, \eta, P)$ and let $(\eta_t)_{t \in T}$ be the filtration generated by $W$. 

A functional of the Brownian motion of the form

$$F = f(W(t_1), ..., W(t_n)) \quad (1.1)$$

with $t_1, ..., t_n \in T$ and $f \in C^\infty_b(R^n)$ is called a smooth random variable and this class is denoted by $S$. ($C^\infty_b$ is the set of infinitely differentiable function on $[0,b]$ whose derivatives of any order are null at $b$)

The Malliavin derivative is defined on $S$ as

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(t_1), ..., W(t_n))1_{[0,t_i]}(t), \quad t \in T,$$
if $F$ has form (1.1). The operator $D$ is closable and it can be extended to the closure of $S$ with respect to the seminorm

$$
\| F \|_{k,p}^p = E|F|^p + \sum_{j=1}^k E \| D^{(j)}F \|_{L^2(T)}^p
$$

where $D^{(i)}$ denotes the $i$th iterated derivative. Note that if $F$ is $\eta_A$-measurable ($A$ being a Borel subset of $\mathbb{R}$), then $DF = 0$ on $A^c \times \Omega$. The adjoint of $D$ is denoted by $\delta$ and it is called the Skorohod integral (see [1], [2], [3], [4], [5]). That is $\delta$ is defined on its domain $\text{Dom}(\delta) = \{ u \in L^2(T) \times \Omega / |E \int_0^1 u_s D_s F ds| \leq c \| F \|_{L^2(\Omega)} \}$ and it is given by duality relationship

$$
E(\delta^2(u)) = E \int_0^1 u_s D_s F ds, \quad u \in \text{Dom}(\delta), \quad F \in S.
$$

Recall the variance of the Skorohod integral is

$$
E(\delta^2(u)) = E \int_0^1 u_s^2 d\theta + E \int_0^1 \int_0^1 D_\gamma u_\theta D_\theta u_\gamma d\theta d\gamma, \quad (1.2)
$$

and the commutative relationship between the derivative operator and the Skorohod integral: if $u \in \text{Dom}(\delta)$ with $D_t u \in \text{Dom}(\delta)$, then

$$
D_t \delta(u) = u_t + \delta(D_t u), \quad t \in T. \quad (1.4)
$$

By $\ell^{k,p}$ we denote the set $L^2(T; M^{k,p})$, for $k \geq 1$ and $p \geq 2$. Note that $\ell^{k,p}$ is a subset of the domain of $\delta$. Meyer’s inequalities implies

$$
E|\delta|^p \leq \|u\|_{1,p}^p. \quad (1.5)
$$

The following generalized Ocone-Clark formula was given in Nualart and Pardoux.

$$
F = E(F/\eta_{[s,t]}^c) + \int_s^t E(D_\theta F/\eta_{[\theta,t]}^c) dW_\theta, \quad \text{for } F \in M^{1,2}. \quad (1.6)
$$

($\eta_{[s,t]}^c$ denotes the $\sigma$-algebra generated by the increments of the Wiener process $W$ on $T \setminus [s, t]$)

It holds that, if the process $u \in \ell^{1,2}$, then $u1_{[0,t]}$ belongs to $\text{Dom}(\delta)$ for every $t$ and we can consider the indefinite Skorohod integral

$$
X_t = \delta(u1_{[0,t]}) = \int_0^t u_s dW_s.
$$
Let us define, for \( k \geq 1 \) and \( p \geq 2 \), the sets of processes

\[
N^{k,p} = \{ X = (X_t)_{t \in T}, X_t = \int_0^t u_s dW_s, u \in \ell^{k,p} \},
\]

\[
\rho^{k,p} = \{ Y = (Y_t)_{t \in T}, Y_t = \int_0^t E[v_s/\eta_{[s,t]}] dW_s, v \in \ell^{k,p} \}.
\]

We will refer to the elements of \( \rho^{k,p} \) as Ito-Skorohod integral processes and to the elements of \( N^{k,p} \) as Skorohod integral processes \([6]\).

As a consequence, to study Skorohod integral processes it suffices to study Ito-Skorohod integral processes, which have two interesting properties. Firstly, note that the integral

\[
Y_t = \int_0^t E[u_\theta/\eta_{[\theta,t]}] dW_\theta
\]

exists even for \( u \in L^2(T \times \Omega) \) and it has similarities with a classical Ito integral. Observe that this integral is an isometry in the sense

\[
(E \int_0^t E[u_\theta/\eta_{[\theta,t]}] dW_\theta)^2 = E \int_0^t (E[u_\theta/\eta_{[\theta,t]}])^2 d\theta.
\]

Secondly, if we define for every \( \lambda \leq t \),

\[
Y^\lambda_t = \int_0^\lambda E[u_\theta/\eta_{[\theta,t]}] dW_\theta
\]

then the process \((Y^\lambda_t)_{\lambda \leq t}\) is a \( \eta_{[\lambda,t]} \)-martingale and we have \( \lim_{\lambda \to t, \lambda \leq t} Y^\lambda_t = Y_t \) almost surely and in \( L^2 \).

We will define now the stochastic integral with respect to Ito-Skorohod integral processes.

**Definition.** Let \( u, v \in L^2(T \times \Omega) \) be adapted processes and consider more

\[
Y_t = Y_0 + \int_0^t E[u_\theta/\eta_{[\theta,t]}] dW_\theta + \int_0^t E[v_\theta/\eta_{[\theta,t]}] d\theta.
\]

We put by definition, for any adapted square integrable processes X,

\[
\int_0^t X_s dY^s := \int_0^t X_s dY^s_t \quad (1.7)
\]

where

\[
Y^\lambda_t = Y_0 + \int_0^\lambda E[u_\theta/\eta_{[\theta,t]}] dW_\theta + \int_0^\lambda E[v_\theta/\eta_{[\theta,t]}] d\theta.
\]

and the integral in the right side of (1.7) is understood in the semimartingale sense.
2 Ito-Skorohod stochastic equations

In this section we state and prove an existence and uniqueness theorem for a class of anticipating fractional stochastic integral equations using the method of Picard’s iterations. It is know that, in the anticipating stochastic calculus this method cannot be applied because the formula of the mean square of the Skorohod integral involves the Malliavian derivative and one cannot find closed formula. We define here a new class of anticipating equations, located between Ito and Skorohod equations, that can be classical techniques.

Consider the following fractional stochastic integral equation

\[ X_t = Z + \frac{1}{\Gamma(\alpha)} \int_0^t a(t, s, E(X_s/\eta_{[s,t]})) \frac{dW_s}{(t-s)^{(1-\alpha)}} + \frac{1}{\Gamma(\alpha)} \int_0^t b(t, s, X_s) \frac{dW_s}{(t-s)^{(1-\alpha)}} ds, \quad t \in T \]  \hspace{1cm} (2.1)

where \( 0 < \alpha < 1 \) and \( a(t, s, x), b(t, s, x) : \{ t, s; 0 \leq s \leq t \} \times R \rightarrow R \).

Note that the stochastic integral from above is a Skorohod integral since the integrand is not adapted and the initial condition is anticipating.

The coefficients \( a(t, s, x) \) and \( b(t, s, x) \) satisfy the following conditions:

**H1 : (Measurability)**: \( a \) and \( b \) are jointly measurable in 

\[ (t, s, x) \in \{ t, s ; 0 \leq s \leq t \} \times R. \]

**H2 : (Lipschitz condition)**: There exist a constant \( D > 0 \) such that

\[ |a(t, s, x) - a(t, s, y)| + |b(t, s, x) - b(t, s, y)| \leq D|x - y|. \hspace{1cm} (2.2) \]

**H3 : (Linear growth condition)**: There exist a constant \( k > 0 \) such that

\[ |a(t, s, x)|^2 + |b(t, s, x)|^2 \leq k^2(1 + |x|^2). \hspace{1cm} (2.3) \]

**H4 :** \( Z \) is a random variable with \( E|Z| < \infty \).

Again let \( 1 - \alpha = \frac{\beta}{2} \), \( 0 < \beta < 1 \).

A square integrable process that satisfies almost surely (2.1) is called a solution of equation (2.1). For given coefficients \( a \) and \( b \), any solution is unique if, for every \( t \in T \),

\[ P(X^1_t = X^2_t) = 1 \] for any two solution \( X^1 \) and \( X^2 \) with the same initial condition.

**Theorem.** Under assumptions H1 - H4, stochastic equation (2.1) has a unique solution \( X_t \) on \( T \) with

\[ \sup_{0 \leq t \leq 1} E|X_t|^2 < \infty. \hspace{1cm} (2.4) \]
Proof: Throughout this proof, K will denote a generic constant depending on k and $E|Z|^2$.

Let us consider the usual Picard iterations $X_t^{(0)} = Z$ and

$$X_t^{(n+1)} = Z + \frac{1}{\Gamma(\alpha)} \int_0^t a(t, s, E(X_s^{(n)}/\eta_{s,t}[c])) \frac{1}{(t-s)^{1-\alpha}} dW_s + \frac{1}{\Gamma(\alpha)} \int_0^t b(t, s, X_s^{(n)}) (t-s)^{1-\alpha} ds. \quad (2.5)$$

We first prove the existence of the solution. We have from H3 that

$$E|X_t^{(1)} - X_t^{(0)}|^2 \leq \frac{2}{(\Gamma(\alpha))^2} \{ E \int_0^t \frac{a(t, s, E(Z/\eta_{s,t}[c]))}{(t-s)^{2/3}} dW_s \}^2 + E \int_0^t \frac{b(t, s, Z)}{(t-s)^{2/3}} ds \}^2$$

$$\leq \frac{2k^2}{(\Gamma(\alpha))^2} \{ \int_0^t \frac{(1 + |E(Z/\eta_{s,t}[c])|^2)}{(t-s)^{2/3}} ds + \int_0^t \frac{(1 + E|Z|^2)}{(t-s)^{2/3}} ds \} \leq \frac{Kt^{1-\beta}}{1 - \beta}.$$

Using the same arguments and condition H4, we obtain

$$E|X_t^{(n+1)} - X_t^{(n)}|^2 \leq \frac{2}{(\Gamma(\alpha))^2} \{ E \int_0^t \frac{a(t, s, E(X_s^{(n)}/\eta_{s,t}[c])) - a(t, s, E(X_s^{(n-1)}/\eta_{s,t}[c]))}{(t-s)^{2/3}} dW_s \}^2$$

$$+ E \int_0^t \frac{|b(t, s, X_s^{(n)}) - b(t, s, X_s^{(n-1)})|}{(t-s)^{2/3}} ds \}^2$$

$$\leq \frac{2}{(\Gamma(\alpha))^2} \{ E \int_0^t \frac{a(t, s, E(X_s^{(n)}/\eta_{s,t}[c])) - a(t, s, E(X_s^{(n-1)}/\eta_{s,t}[c]))}{(t-s)^{2/3}} ds \}^2$$

$$+ E \int_0^t \frac{|b(t, s, X_s^{(n)}) - b(t, s, X_s^{(n-1)})|}{(t-s)^{2/3}} ds \} \leq \frac{K^{n+1}t^{(n+1)-\beta}}{(1 - \beta)(2 - \beta)(3 - \beta)\ldots((n + 1) - \beta)} \quad (2.6)$$

Relation (2.6) and standard arguments imply the convergence in $L^2(\Omega)$ on the successive approximations $X_t^{(n)}$ to a limit $X_t$ defined by

$$X_t = Z + \sum_{n=0}^{\infty} (X_t^{(n+1)} - X_t^{(n)}).$$

To prove that X is a solution, we take $L^2(\Omega)$-limit in (2.5) as $n \to \infty$. Obviously,

$$\frac{2}{(\Gamma(\alpha))^2} E \int_0^t \frac{a(t, s, E(X_s^{(n)}/\eta_{s,t}[c])) - a(t, s, E(X_s/\eta_{s,t}[c]))}{(t-s)^{2/3}} dW_s \}^2$$
\[ \leq K \int_0^t \frac{E|X_s^{(n)} - X_s|^2}{(t-s)^\beta} ds \to 0 \]

as \( n \to \infty \) and
\[ \frac{2}{(\Gamma(\alpha))^2} E\left| \int_0^t \left( b(t, s, X_s^{(n)}) - b(t, s, X_s) \right) ds \right|^2 \leq K \int_0^t \frac{E|X_s^{(n)} - X_s|^2}{(t-s)^\beta} ds \to 0 \]

as \( n \to \infty \).

To prove the uniqueness, for any two solution \( X, Y \) with the same initial condition and for every \( t \in T \) we have
\[ E|X_t - Y_t|^2 \leq K \int_0^t \frac{E|X_s - Y_s|^2}{(t-s)^\beta} ds. \quad (2.7) \]

Set \( E|X_t - Y_t|^2 = \gamma_t \). It easy to see that
\[ \gamma_t \leq K \int_0^t \frac{\gamma_s}{(t-s)^\beta} ds \quad (2.8) \]

where \( K \) is a positive constant. Thus, (see [7], [8], [9]),
\[ \gamma_t \leq \frac{K^2(\Gamma(1-\beta))^2}{\Gamma(2(1-\beta))} \int_0^t \frac{\gamma_s}{(t-s)^{2\beta-1}} ds, \]

and, by a classical argument, we get
\[ \gamma_t \leq \frac{K^n(\Gamma(1-\beta))^n}{\Gamma(n(1-\beta))} \int_0^t \frac{\gamma_s}{(t-s)^{n\beta-(n-1)}} ds. \quad (2.9) \]

Thus for sufficiently large \( n, n > \frac{1}{1-\beta} \), we get from (2.9)
\[ \gamma_t \leq \frac{C^n}{\Gamma(n(1-\beta))}, \quad (2.10) \]

where \( C \) is a positive constant. Taking limit as \( n \to \infty \), we find that (2.10) leads to \( \gamma_t = 0 \).

Equation (2.1) has many important financial applications (see [10], [11]). The fractional Black-Scholes market consists of a bank account or a bond and a stock. The prices process \( A_t \) of the bond at time \( t \) is given by \( A_t = e^{\int_0^t r(s) ds} \), where \( r(s) \geq 0, s \in [0, t] \), is the interest rate. A portfolio is a pair \((u_t, v_t)\) of random variables for fixed \( t \in [0, 1] \). The price \( X_t \) of the stock could be governed by a fractional stochastic integral equation of the form
\[ X_t = X_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\sigma(s)E(X_s/\eta_{s,\xi})}{(t-s)^{1-\alpha}} dW_s + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\mu(s)X_s}{(t-s)^{1-\alpha}} ds, \]
Here the drift $\mu \geq 0$ and volatility $\sigma \geq 0$ are continuous functions on $[0,1]$. The numbers $u_t$ and $v_t$ are the bond and stock units, respectively (held by an investor). Hence the corresponding value process is $V_t = u_t A_t + v_t X_t$. The process $V_t$ could be governed by the equation

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\sigma(s) v_s E(X_s/\eta_{[s,t]}^c)}{(t-s)^{1-\alpha}} dW_s$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t r(s) A_s u_s \frac{1}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \mu(s) X_s v_s \frac{1}{(t-s)^{1-\alpha}} ds.$$

**References**


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