Asymptotic Behavior of Traveling Wave Fronts of Lotka-Volterra Competitive System

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Abstract
This paper is concerned with the asymptotic behavior of the traveling wave fronts in Lotka-Volterra competition system. By Laplacian transform, we prove that the traveling wave fronts of the system grow exponentially when the traveling coordinate tends to negative infinity.

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1 Introduction
In this paper, we are concerned with the asymptotic behavior of the traveling wave fronts of the following competition-diffusion systems [6, 8]

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= d_1 \Delta u(x,t) + r_1 u(x,t) (1 - a_1 u(x,t) - b_1 v(x,t)), \\
\frac{\partial v(x,t)}{\partial t} &= d_2 \Delta v(x,t) + r_2 v(x,t) (1 - a_2 v(x,t) - b_2 u(x,t)),
\end{align*}
\]

(1.1)

where \( x \in \mathbb{R} \) and \( t > 0 \), \( u(x,t) \) and \( v(x,t) \) represent the population densities of two competition species, and all the parameters are positive. It is clear that model (1.1) has a trivial (no species) equilibrium \( E_0 = (0, 0) \), and a positive (two coexisting species) spatial homogeneous one

\[
E^* = \left( \frac{a_2 - b_1}{a_1 a_2 - b_1 b_2}, \frac{a_1 - b_2}{a_1 a_2 - b_1 b_2} \right)
\]

provided that with \( a_2 > (\leq)b_1 \) and \( a_1 > (\leq)b_2 \).

For (1.1) and its traveling wave front, there have been many interesting and important papers, we refer to Conley and Gardner [2], Gardner [3], Li et al. [4], Tang and Fife [6], van Vuuren [8] and the references cited therein.
If \( a_2 > b_1 \) and \( a_1 > b_2 \), then
\[
\begin{aligned}
\frac{du_1(t)}{dt} &= r_1u_1(t)\left[1 - a_1u_1(t) - b_1u_2(t)\right], \\
\frac{du_2(t)}{dt} &= r_2u_2(t)\left[1 - a_2u_2(t) - b_2u_1(t)\right]
\end{aligned}
\] (1.2)
is monostable \((E^* \text{ is stable})\). It is well known that the asymptotic behavior of the monostable wave front is very important in considering the further properties of the traveling wave fronts [7], such as the stability and the uniqueness, since this often determines the choice of perturbation space. Thus, it is necessary to understand the asymptotic behavior of traveling wave fronts near the unstable equilibrium \( E_0 \) when the traveling wave fronts connecting \( E_0 \) with \( E^* \) are concerned. This constitutes the purpose of this paper.

In this paper, by applying the Laplace transform, we prove that the traveling wavefronts of (1.1) grow exponentially when the traveling wave coordinate tends to negative infinity. Such a method was earlier used in Carr and Chmaj [1] for the nonlocal diffusion equations, and recently in Pan [5] for the Fisher equation with delay.

2 Main Result

It is well known that the traveling wave front of (1.1) is a special solution
\[
(u(x, t), v(x, t)) = (\phi(x + ct), \psi(x + ct)),
\]
where \( c > 0 \) is the wave speed and \( \phi, \psi \) are the wave profile functions which are monotone. If we replace \( x + ct \) by \( t \), then \( (\phi, \psi) \) must satisfy the following differential system [6, 8]
\[
\begin{aligned}
d_1\phi''(t) - c\phi'(t) + r_1\phi(t)[1 - a_1\phi(t) - b_1\psi(t)] = 0, \\
d_2\psi''(t) - c\psi'(t) + r_2\psi(t)[1 - a_2\psi(t) - b_2\phi(t)] = 0,
\end{aligned}
\] (2.1)
and we are interested in the following asymptotic boundary behaviors
\[
\lim_{t \to -\infty} (\phi(t), \psi(t)) = (0, 0), \quad \lim_{t \to +\infty} (\phi(t), \psi(t)) = (k_1, k_2). \tag{2.2}
\]
We first recall the existence of traveling wave fronts of equation (1.1), which was established by Tang and Fife [6], van Vuuren [8].

**Theorem 2.1** (1.1) has a traveling wave front, namely, (2.1) has a monotone solution satisfying (2.2), if and only if \( c \geq c^* = 2 \max\{\sqrt{d_1r_1}, \sqrt{d_2r_2}\} \).

By the monotonicity of traveling wave fronts, the following result is clear.

**Lemma 2.2** \( \lim_{t \to \pm \infty} (\phi'(\xi), \psi'(\xi)) = (0, 0) \) holds.
Lemma 2.3. For any $t < 0$,

$$0 < \int_{-\infty}^{t} \phi(\xi) d\xi < +\infty, \quad 0 < \int_{-\infty}^{t} \int_{-\infty}^{s} \phi(\xi) d\xi ds < +\infty,$$

$$0 < \int_{-\infty}^{t} \psi(\xi) d\xi < +\infty, \quad 0 < \int_{-\infty}^{t} \int_{-\infty}^{s} \psi(\xi) d\xi ds < +\infty.$$

Proof. By (2.1) and (2.2), there exist $\xi' < 0$ and $0 < \epsilon < r_{1}$ such that

$$-d_{1}\phi''(\xi) + c\phi'(\xi) \geq (r_{1} - \epsilon)\phi(\xi), \quad \xi < \xi'.$$

Then the boundedness and monotonicity of $\phi(\xi)$ imply that

$$0 < (r_{1} - \epsilon) \int_{-\infty}^{\xi} \phi(t) dt \leq -d_{1}\phi'(\xi) + c\phi(\xi) < +\infty \text{ for } \xi < \xi', \quad (2.3)$$

and (2.3) also indicates that

$$0 < \int_{-\infty}^{t} \int_{-\infty}^{s} \phi(\xi) d\xi ds < +\infty \text{ for } t < \xi'.$$

By the boundedness of $\phi(\xi)$ with $\xi \in [\xi', 0]$, then

$$0 < \int_{-\infty}^{t} \phi(\xi) d\xi < +\infty \text{ and } 0 < \int_{-\infty}^{t} \int_{-\infty}^{s} \phi(\xi) d\xi ds < +\infty \text{ for any } t < 0.$$

Similarly, we can prove that

$$0 < \int_{-\infty}^{t} \psi(\xi) d\xi < +\infty \text{ and } 0 < \int_{-\infty}^{t} \int_{-\infty}^{s} \psi(\xi) d\xi ds < +\infty \text{ for } t < 0.$$

The proof is complete. \qed

Lemma 2.4. There exist constants $\lambda' > 0$ and $\lambda'' > 0$ such that $\sup_{\xi \in \mathbb{R}} \phi(\xi) e^{\lambda' \xi} < \infty$ and $\sup_{\xi \in \mathbb{R}} \psi(\xi) e^{\lambda'' \xi} < \infty$.

Proof. Let $\omega(t) = \int_{-\infty}^{t} \phi(\xi) d\xi$, then Lemma 2.3 implies that $\omega(t) < \infty$ for any $t < 0$. On the other hand, (2.3) also indicates that for any $r > 0$ and $t < \xi'$

$$c\omega(t) \geq (r_{1} - \epsilon) \int_{-\infty}^{t} \omega(\theta) d\theta \geq (r_{1} - \epsilon) \int_{t-r}^{t} \omega(\theta) d\theta \geq r(r_{1} - \epsilon)\omega(t - r).$$
Thus, there exists $r_0 > 0$ and some $\theta \in (0, 1)$ such that $\omega(t - r) \leq \theta \omega(t)$ for any $t < \xi'$. Let $e(t) = \omega(t)e^{-\lambda t}$ with $\lambda = \frac{1}{r_0} \ln \frac{1}{\theta}$. Then
\[
e(t - r_0) = \omega(t - r_0)e^{-\lambda(t - r_0)} = \omega(t - r_0)e^{-\lambda t + \lambda r_0}
= \frac{1}{\theta}\omega(t - r_0)e^{-rt} \leq e^{-rt}\omega(t) = e(t), t < \xi',
\]
which implies that $e(t)$ is nonincreasing for all $t < \xi'$. We also note that $\lim_{\xi \to -\infty} \omega(\xi)e^{-\lambda \xi} = 0$ and $\omega(\xi)e^{-\lambda \xi}$ is bounded, then $\sup_{\xi \in R} \omega(\xi)e^{\lambda \xi} < \infty$.

In addition, for $\xi < \xi'$, we have $0 < d_1 \phi(\xi) < c \omega(\xi)$, then $\sup_{\xi \leq \xi'} \phi(\xi)e^{-\lambda \xi} < +\infty$ is clear. By the boundedness of $\phi(\xi)$, we know that $\sup_{\xi \in R} \phi(\xi)e^{-\lambda \xi} < +\infty$ holds.

Similarly, there exists a constant $\lambda'' > 0$ such that $\sup_{\xi \in R} \phi(\xi)e^{\lambda'' \xi} < +\infty$ is true. The proof is complete. \hfill \Box

**Remark 2.5** By the boundedness of the traveling wave front, it is clear that
\[
\int_{-\infty}^{\infty} \phi(\xi)e^{\lambda \xi} d\xi < \infty \text{ for all } \lambda \in (0, \lambda'), \quad \int_{-\infty}^{\infty} \psi(\xi)e^{\lambda \xi} d\xi < \infty \text{ for all } \lambda \in (0, \lambda'').
\]

**Lemma 2.6** Let $F(\lambda) = \int_0^\infty u(x)e^{\lambda x} dx$ with $u(x)$ being a positive decreasing function. Assume that $F$ has the representation
\[
F(\lambda) = \frac{H(\lambda)}{(\lambda + \alpha)^{k+1}},
\]
where $k > -1$ and $H$ is analytic in the strip $0 < \text{Re} \lambda \leq \alpha$. Then
\[
\lim_{x \to -\infty} \frac{u(x)}{x^k e^{-\alpha x}} = \frac{H(-\alpha)}{T(\alpha + 1)}.
\]

**Remark 2.7** Lemma 2.6 is another version of the Ikehara’s Theorem, see this by Carr and Chmaj [1] and the proof is omitted.

For $c > c^* = 2 \max\{\sqrt{d_1 r_1}, \sqrt{d_2 r_2}\}$, define constants $\lambda_1$ and $\lambda_2$ as follows
\[
\lambda_1 = \frac{c - \sqrt{c^2 - 4d_1 r_1}}{2d_1}, \quad \lambda_2 = \frac{c - \sqrt{c^2 - 4d_2 r_2}}{2d_2}.
\]

**Theorem 2.8** Assume that $(\phi(t), \psi(t))$ is the traveling wave front of (1.1) and $c > c^*$ holds. Then there exist two constants $h_1, h_2$ such that
\[
\lim_{t \to -\infty} (\phi(t)e^{-\lambda_1 (t+h_1)}, \psi(t)e^{-\lambda_2 (t+h_2)}) = (1, 1).
\]
Proof. For $\lambda > 0$, define two-side Laplace transforms as follows

$$L_1(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} \phi(t) dt, \quad L_2(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} \psi(t) dt.$$  

Then (2.1) implies that

$$\left(d_1\lambda^2 - c\lambda + r_1\right) L_1(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} \left[a_1\phi^2(t) + b_1\phi(t)\psi(t)\right] dt, \quad (2.4)$$

$$\left(d_2\lambda^2 - c\lambda + r_2\right) L_2(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} \left[a_2\psi^2(t) + b_2\phi(t)\psi(t)\right] dt \quad (2.5)$$

by the properties of Laplacian transform. It is clear that (2.4) is well defined for $\lambda \in (0, \lambda')$ and (2.5) is well defined for $\lambda \in (0, \lambda'')$, respectively.

For (2.4), if the left side is well defined for all $\lambda \in (0, \gamma_1)$ with some $\gamma_1 > 0$, then the right side is well defined for all $\lambda \in (0, \min\{2\gamma_1, \gamma_1 + \lambda''\})$. Hence (2.4) is well defined for all $\lambda > 0$ or for some bounded interval related to the zeros of $d_1\lambda^2 - c\lambda + r_1 = 0$. However, since $c > c^*$ holds, then the first case is impossible since the right side of (2.4) is positive for all admissible $\lambda$ while the left side change sign at $\lambda = \lambda_1$. Then Remark 2.5 and Lemma 2.6 imply that $\lim_{t \to -\infty} \phi(t)e^{-\lambda_1(t+h_1)} = 1$ holds for some $h_1 \in \mathbb{R}$.

In a similar way, we can prove that $\lim_{t \to -\infty} \psi(t)e^{-\lambda_2(t+h_2)} = 1$ holds for some $h_2 \in \mathbb{R}$. The proof is complete.  

References


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