Travelling Waves Solutions
for Ice-Sheets Equation

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Abstract. In this paper, we study the travelling waves solutions for the following equation

$$u_t = \left(u^n |u_x|^{p-2} u_x\right)_x - f(u), \ x \in \mathbb{R}, \ t > 0,$$

where $m \geq 0$ and $p > 2$. Existence and asymptotic behavior are established.

1. Introduction

This paper is concerned with the nonlinear equation

$$u_t = \left(u^n |u_x|^{p-2} u_x\right)_x - f(u), \ x \in \mathbb{R}, \ t > 0,$$

where $f \in C([0, +\infty[ , \mathbb{R})$. Equation (1.1.1) is a simple model of the equation which intervenes in several field. For $p = 2$, we obtain an equation of the porous medium which is a simple model of equation appearing in several areas like fluid dynamics, combustion theory, soil physics and reaction chemistry [2, 3, 4, 5, 6, 7]. Many other related references can be found in [8, 9, 10, 13, 23]. In particular, it models heat transfer with thermal conductivity and sources or sinks of thermal energy depending on the temperature [27, 28, 29, 30].

If $p > 2$, Equation (1.1.1) intervenes in the description of ice sheet dynamics. See [14, 19, 20, 21].

We are interested in a particular class of solutions to (1.1.1), the travelling waves. By a travelling wave solution (TW) with velocity $c \in \mathbb{R}$ we mean a solution $u(x, t)$ of (1.1.1) in $D = \{(x, t) : -\infty < x < +\infty, t > 0\}$ of the form

$$u(x, t) = \varphi(ct - x),$$

where $c \in \mathbb{R}$ and $\varphi$ is a positive function, with $\varphi^m |\varphi'|^{p-2} \varphi'$ and $\varphi$ is continuous.

We say that $u(x, t) = \varphi(ct - x)$ is a finite travelling wave (FTW) if $\varphi$ is only defined in an interval of the form $(\xi_0, \infty)$ with $\varphi(\xi_0) = 0$ and $\varphi > 0$ in $(\xi_0, \infty)$. In addition, if $\xi_0 = -\infty$, then $\varphi$ is said a global travelling wave solution. The importance of special solutions in TW form to equations lies in the fact that
sometimes they give insight into the behavior of some classes of solutions of the same equation with arbitrary initial conditions. In particular, the property of finite propagation and the local behavior near the corresponding interface is almost always characterized by comparison with finite TW, by which we mean travelling waves with support bounded from one side. Many particular cases of Equation (1.1.1) have been considered in detail in the literature. For $m \geq 1$ and $p = 2$ the travelling waves solutions of Equation (1.1.1) were analyzed by several authors, for $f(u) = \pm u^q$ we refer to [22, 23, 24], for $f(u) = \pm u^q(1 - u)$ see [11, 12, 15, 16, 17, 18] and for Fisher-KPP equation with general exponents corresponding to $f(u) = u^q - u^p$ see [26]. Our goal here is the study of the finite travelling waves and also the global travelling waves satisfy

\begin{equation}
\lim_{\xi \to -\infty} \varphi(\xi) = 0.
\end{equation}

i.e. waves with the condition of vanishing at $-\infty$.

Throughout this work we consider the case $p > 2$ and $m \geq 0$. Moreover, we make the following assumptions :

\begin{enumerate}
\item[(h\textsubscript{0})] $f(u) > 0$, for all $u \neq 0$
\item[(h\textsubscript{1})] there exists $q > 0$ such that
\begin{equation*}
f(x) = o(x^0) \quad \text{when } x \to 0,
\end{equation*}
\item[(h\textsubscript{2})] there exists $n > 0$ such that
\begin{equation*}
f(x) = o(x^n) \quad \text{when } x \to \infty.
\end{equation*}
\end{enumerate}

We define $k(s, p, m) = \{(s + 1)(p - 1) + m\}/p$, $s > 0$ and $f \sim g$ if $\lim_{x \to g} f(x) \in \mathbb{R}$. The main results are the following

**Theorem 1.1.** Equation (1.1.1) admits FTW solutions with velocity $c \in \mathbb{R}$ if and only if one of the following conditions is satisfied:

(i) $c < 0$, $q < 1$ and $n \leq m + p - 1$.
(ii) $c > 0$ and $n \leq m + p - 1$.

**Theorem 1.2.** Equation (1.1.1) admits a global TW solutions if and only if $c < 0$, $q \geq 1$ and $n \leq m + p - 1$.

**Theorem 1.3.** Let $\varphi$ a FTW solution of Equation (1.1.1) with velocity $c \neq 0$. Then $\varphi(\xi) \sim \xi^s$ when $\xi$ goes to 0, with for $c > 0$ or $(k(q, p, m) - 1)c > 0$, $s = (p - 1)/(p + m - \inf \{k(q, p, m), 1\} - 1)(m + p - 1)$, for $c < 0$ and $k(q, p, m) > 1$, $s = 1/(1 - q)$.

**Theorem 1.4.** Let $\varphi$ a FTW solution of Equation (1.1.1) with velocity $c$. Then the following properties hold

(a) For $c > 0$ or $(k(n, p, m) - 1)c < 0$, we have :
(i) if \( k(n, p, m) = m + p - 1 \), then \( \log(\phi^{k(n,p,m)/(p-1)}(\xi)) \sim \xi \) when \( \xi \) goes to \( \infty \).
(ii) if \( k(n, p, m) < m + p - 1 \), then \( \phi(\xi) \sim \xi^s \) when \( \xi \) goes to \( \infty \), with \( s = (p-1)/(p + m - \sup \{ k(n,p,m), 1 \} - 1) \).

(b) For \( c < 0 \) and \( k(n, p, m) < 1 \), \( \phi(\xi) \sim \xi^{1/(1-n)} \) when \( \xi \) goes to \( \infty \).

2. Proof of the theorems 1.1, 1.2, 1.3 and 1.4.

We consider Equation (1.1.1) with \( p > 2 \) and \( m > 0 \). We take \( u(x, t) = \phi(\xi) \) in the equation (1.1.1), we obtain

\[
(2.2.1) \quad c\phi' = (\phi^m |\phi'|^{p-2}\phi')' - f(\phi).
\]

If \( \phi \) is a FTW solution such that \( \phi(\xi_0) = 0 \), by translation \( \xi \to \xi - \xi_0 \), we can suppose

\[
(2.2.2) \quad \phi(0) = 0.
\]

Integrating (2.2.1) we obtain

\[
(2.2.3) \quad \phi^m |\phi'|^{p-2}\phi'(0) = 0.
\]

Consequently, the search for FTW solutions is restricted in the study of the problem: To find \( c \in \mathbb{R} \) and a function \( \phi \neq 0 \) such that

\[
(2.2.4) \left\{ \begin{array}{l}
    c\phi' = (\phi^m |\phi'|^{p-2}\phi')' - f(\phi) \\
    \phi(0) = \phi^m |\phi'|^{p-2}\phi'(0) = 0.
  \end{array} \right.
\]

In addition, if \( \phi \) is global checking the condition (1.1.3), then we integrate the equation (2.2.1) on \( [\xi_1, \xi_2] \) and passing to the limit, we thus get

\[
(2.2.5) \lim_{\xi_1 \to -\infty} \phi^m |\phi'|^{p-2}\phi'(\xi_1) = \phi^m |\phi'|^{p-2}\phi'(\xi_2) - c\phi(\xi_2) - \int_{-\infty}^{\xi_2} f(\phi(s))ds.
\]

Using (1.1.3) and the preceding formula we have

\[
(2.2.6) \lim_{\xi \to -\infty} \phi^m |\phi'|^{p-2}\phi'(\xi) = 0,
\]

Using the definition if \( \phi \) is a FTW and using (1.1.3) if \( \phi \) is a global TW, one can find \( \xi, \xi_0 \in \mathbb{R} \) such that \( \phi \) is strictly increasing on \( [\xi, \xi_0] \), since \( \phi \) cannot have a local maximum, then \( \phi \) is strictly increasing on \( \{ \phi > 0 \} \). Consequently, the research of the global TW solutions is reduced to the resolution of the problem : To find \( c \in \mathbb{R} \) and a function \( \phi \neq 0 \) which check

\[
(2.2.7) \left\{ \begin{array}{l}
    c\phi' = (\phi^m (\phi')^{p-1})' - f(\phi), \\
    \lim_{\xi \to -\infty} \phi(\xi) = \lim_{\xi \to -\infty} \phi^m (\phi')^{p-1} = 0.
  \end{array} \right.
\]

We transform Equation (2.2.1) into a first order system, by introducing the variables

\[
(2.2.8) \quad x = \phi \text{ et } y = \phi^m (\phi')^{p-1},
\]
This involves that Equation (2.2.1) is equivalent to the system

\[
(S) \quad \begin{cases}
\frac{dx}{d\xi} = x^{-m/(p-1)}y^{1/(p-1)}, \\
\frac{dy}{d\xi} = f(x) + cx^{-m/(p-1)}y^{1/(p-1)}.
\end{cases}
\]

where \( m \geq 0 \) and \( p > 2 \). The variables satisfy \( \xi \in \mathbb{R}, \lim_{\xi \to -\infty} x(\xi) = 0 \), and if \( x(\xi) = 0 \) for \( \xi \leq 0 \) the conditions (2.2.3) and (2.2.6) become \( \lim_{x \to 0} y(x) = 0 \).

Now we consider the Cauchy problem associated with the vector field given as follows

\[
(Q) \quad \begin{cases}
\frac{dy}{dx} = c + f(x)x^{m/(p-1)}y^{-1/(p-1)}, \\
y(0) = 0.
\end{cases}
\]

In order to find the solution of the problem (2.2.10), we consider the problem

\[
(Q_\varepsilon) \quad \begin{cases}
\frac{dy}{dx} = c + f(x)x^{m/(p-1)}y^{-1/(p-1)} = F(x,y), \\
y(0) = \varepsilon,
\end{cases}
\]

where \( \varepsilon \in \mathbb{R}_+^* \).

**Lemma 2.1.** For each \( \varepsilon > 0 \), problem \((Q_\varepsilon)\) admits a unique global solution.

**Proof.** As \( f \) is continuous then \( x \to F(x,y) \) is continuous. Thus, for \( x \) fixed, the function \( F(x,.,:) \) is locally lipchitzien on \( \mathbb{R}_+^* \), from where according to the theory of ODE [1], there exists a local unique solution of the problem \((Q_\varepsilon)\) noted \( y_\varepsilon \). If \( c > 0 \), the function \( x \to y_\varepsilon(x) \) is strictly increasing, thus by using the fact that \( f \) is positive and formula (2.2.12) one obtains

\[
\frac{dy_\varepsilon}{dx} \leq c + f(x)x^{m/(p-1)}\varepsilon^{-1/(p-1)};
\]

this inequality with the continuity of \( f \) implies that \( y_\varepsilon \) is bounded in all bounded and consequently it is global.

In addition, if \( c < 0 \), we introduce the curve \((C')\) solution of the equation \( F(x,y) = 0 \), for \( y > 0 \). Moreover, it is given by the formula

\[
\bar{y}(x) = \left( -\frac{f(x)}{c} \right)^{p-1} x^m
\]

The curve \((C')\) divides the phase-plan \((x,y)\) into two areas \((cf \, figure\, 1)\),

\[
\{(x,y), \bar{y} < y\} \quad \text{and} \quad \{(x,y), y < \bar{y}\}.
\]

Since \( y_\varepsilon(0) = \varepsilon > \bar{y}(0) = 0 \), then necessarily the function \( x \to y_\varepsilon(x) \) is
decreasing and it will become increasing as soon as it touches the curve \((C)\); consequently, there exists \(x_0 > 0\) such that
\[
m_\varepsilon = \min y_\varepsilon = \left(-\frac{f(x_0)}{c}\right)^{p-1} x_0^m > 0,
\]
which implies that
\[
\frac{dy_\varepsilon}{dx} \leq c + f(x)x^{m/(p-1)}m_\varepsilon^{1/(p-1)},
\]
from where \(y_\varepsilon\) is global. 

\[\square\] 

**Lemma 2.2.** For each \(c \in \mathbb{R}_+\), Problem (2.2.10) admits a unique global solution.

**Proof. Uniqueness.** we proceeds by the absurdity. we suppose that there are two solutions \(y\) and \(z\) of the problem (2.2.10) such that \(y \neq z\). Let \(R\) be given by
\[
R = \max \{r > 0, \ z(x) = y(x) \text{ for } 0 \leq x < r\};
\]
it is clear that the indicated number is well defined. Let \(x_0 > R\) such that \(z(x_0) < y(x_0)\) and set
\[
h(x) = (y - z)(x).
\]
Then \(h(R) = 0\) and there exists \(\theta \in [R, x_0[\) such that
\[
0 \leq h(x_0) - h(R) = f(\theta)\theta^{m/(p-1)} \left[ y(\theta)^{-1/(p-1)} - z(\theta)^{-1/(p-1)} \right] < 0.
\]
Which is absurd.

**Existence.** Since \((y_\varepsilon)_{\varepsilon > 0}\) is increasing and \(y_\varepsilon \geq 0\), then it converges to a function \(y\) when \(\varepsilon\) goes 0. We claim that \(y\) is strictly positive. Indeed, let \(\varepsilon > 0\), and we consider the following Cauchy problem
\[
(L_\varepsilon) \quad \begin{cases}
\frac{du}{dt} = \frac{t^{1/(p-1)}}{f(u(t))u^{m/(p-1)}(t) + ct^{1/(p-1)}} = g(t, u), \\
u(0) = \varepsilon.
\end{cases}
\]
It is clear that at a neighborhood of 0,
\[
g(t, u) \sim ct^{1/(p-1)},
\]
then Problem \((L_\epsilon)\) admits a local unique solution noted \(u_\epsilon\).
Moreover if \(c > 0\), we have
\[
0 \leq \frac{du_\epsilon}{dt} \leq \frac{1}{c},
\]
which implies that \(u_\epsilon\) is bounded in all bounded, from where \(u_\epsilon\) is global.
We now turn to the case \(c < 0\), we introduce the curve \((C')\) which is symmetrical to a curve defined by \(F(u, t) = 0\) (\(F(u, t)\) is defined in (2.2.12)) with respect to the axis \(t = u\). We remark that this curve divides the plan-phase into two areas. One of them contains \((0, u_\epsilon(0))\) where it should be noted that \(\frac{du_\epsilon}{dt}\) is positive and tends to \(\infty\) when \(F(t, u_\epsilon(t))\) goes to 0. Consequently \((t, u_\epsilon(t))\) moves away from the curve \((C')\) when \(t \to \infty\). Furthermore, easily we show that \(u_\epsilon\) can not blow up at finite \(t\); which that implies \(u_\epsilon\) is global. In addition, since \(u_\epsilon\) is increasing then \(\lim_{t \to +\infty} u_\epsilon(t) = l\) exists in \([0, +\infty[\). If \(l < +\infty\) then \(\lim_{t \to +\infty} \frac{du_\epsilon}{dt} (u_\epsilon(t)) = 0\). On the other hand, according to equation (2.2.20)
we will have \(\lim_{t \to +\infty} \frac{du_\epsilon}{dt} (u_\epsilon(t)) = \frac{1}{c}\). Which is impossible, thus \(\lim_{t \to +\infty} u_\epsilon(t) = +\infty\). Moreover, \(u_\epsilon\) is one to one of \([0, +\infty[\) with values in \([\epsilon, +\infty[\); we denote by \(z_\epsilon\) the reverse of \(u_\epsilon\). By a simple calculation, we show \(z_\epsilon\) satisfies the Cauchy problem
\[
(2.2.23) \quad \begin{cases} 
\frac{dy}{dx} = c + f(x)x^{m/(p-1)}y^{-1/(p-1)}; \\
y(\epsilon) = 0.
\end{cases}
\]
Thus \(z_\epsilon\) and \(y_\epsilon\) satisfy the same equation on \([\epsilon, +\infty[\) and since \(y_\epsilon(\epsilon) > z_\epsilon(\epsilon) = 0\), it follows that
\[
(2.2.24) \quad y_\epsilon(x) > z_\epsilon(x), \quad \forall \ x \in [\epsilon, +\infty[.
\]
Now, let \(x_0 > 0\) and using the fact that \((z_\epsilon(x))\) is decreasing in order to deduce
\[
(2.2.25) \quad \lim_{\epsilon \to 0} y_\epsilon(x_0) = y(x_0) \geq \lim_{\epsilon \to 0} z_\epsilon(x_0) \geq z_{x_0/3}(x_0) > 0.
\]
therefore \(y\) is strictly positive on \([0, +\infty[\).
To finish one will show that \(y\) is exactly the solution of problem (2.2.10). Since \(y_\epsilon\) is solution of problem \((Q_\epsilon)\) then it is an integral solution, thereby for any function \(\psi \in D([0, +\infty])\) we have
\[
(2.2.26) \quad \int_0^{+\infty} \psi(x) [c + f(x)x^{m/(p-1)}y_\epsilon^{-1/(p-1)}]dx + \int_0^{+\infty} \psi'(x)y_\epsilon(x) = 0.
\]
Using the fact that \((y_\epsilon)_\epsilon\) is decreasing and let us pass at the limit on \(\epsilon\) in the preceding integral form, we obtain
\[
(2.2.27) \quad \frac{dy}{dx} = c + f(x)x^{m/(p-1)}y^{-1/(p-1)}, \quad \text{in } D'[0, +\infty[.
\]
On the other hand \( 0 < y(x) < y_{1/2}(x) \), where \( y_{1/2} \) solution of \( Q_{1/2} \) and since \( y_{1/2} \) is continuous then according to (2.2.27) we obtain \( y \in W^{1,n}([a, b[), \forall n \in \mathbb{N} \). Consequently (2.2.27) is strongly satisfied. Finally, we have \( y_\varepsilon(0) = \varepsilon \to y(0) = 0 \). Which completes the proof of Lemma 2.2.

Let \( A > 0 \) and we consider the following Cauchy problem

\[
\begin{align*}
(2.2.28) & \quad \begin{cases} 
\frac{d}{d\xi}(\phi(\xi)) = \left( \frac{m+p-1}{p-1} \right) y^{1/(p-1)}(\phi^{(p-1)/(m+p-1)})(\xi) = h(\phi(\xi)) \\
\phi(0) = A > 0,
\end{cases}
\end{align*}
\]

where \( y \) is the solution of problem (2.2.10).

Since the function \( y \) is regular and strictly positive on \( [0, +\infty[ \) then \( h \) is also regular and positive; therefore according to the O.D.E theory, problem (2.2.28) admits a unique solution defined on maximal interval \( ]a, b[ \). Since \( \phi \) is increasing then \( \lim_{\xi \to -\infty} \phi(\xi) = l \) exists and positive. If \( -\alpha < \infty \), we can extended \( \phi \) on \( ]-\infty, \alpha[ \) by \( 0 \) if \( l = 0 \). But if \( l > 0 \), we consider the Cauchy problem

\[
(2.2.29) \quad \begin{cases} 
\frac{d}{d\xi}(\phi(\xi)) = h(\phi(\xi)) \\
\phi(0) = l,
\end{cases}
\]

also in this case we can extended \( \phi \) by the restriction of the function solution of Cauchy problem (2.2.29). This contradicts the fact that \( \phi \) is a maximal solution and consequently \( \alpha = -\infty \). We claim that \( \lim_{\xi \to -\infty} \phi(\xi) = 0 \) and \( \lim_{\xi \to -\infty} \phi(\xi) = +\infty \).

In fact, since \( \phi \) is increasing bounded below \( 0 \) then \( \lim_{\xi \to -\infty} \phi(\xi) = 0 \). Indeed, according to (2.2.28) the function \( \phi \) must cross the \( x \)-axis, which contradicts \( \phi \geq 0 \). In other hand, if \( \lim_{\xi \to -\infty} \phi(\xi) = L < +\infty \), necessarily \( \beta = +\infty \) and \( \phi(\xi) < L \), for all \( \xi \in \mathbb{R} \). But, since \( \lim_{\xi \to -\infty} \frac{d}{d\xi}(\phi(\xi)) > 0 \) and \( \phi \) is increasing, then it will cross the axis \( y = L \), which is absurd. From where \( \lim_{\xi \to -\infty} \phi(\xi) = +\infty \).

Setting \( \varphi = \phi^{(p-1)/(m+p-1)} \) and \( \alpha_0 = \inf\{\varphi > 0\} \) and using (2.2.20), by a simple calculation, we can shows that \( \varphi \) satisfies Equation (2.2.1).

Our next goal is to seek the solutions of problem (2.2.28) which are global (\( i.e. \beta = \infty \)) and to see then when-is what they are vanished or they are positive. To this end, we will need of the asymptotic behavior of the solution \( y \) of problem (2.2.10).

**Proposition 2.3.** We suppose that \( c \neq 0, p > 2 \) and \( q > 0 \). Solution \( Y \) behaves as follows:

a) At neighborhood of \( 0 \), \( Y(X) \sim X^{\inf\{k(q,p,m)\}} \) if \( c > 0 \) or \( (k(q,p,m)-1)c \geq 0 \) and \( Y(X) \sim X^{q(p-1)} \) if \( c < 0 \) and \( k(q,p,m) > 1 \).
b) at neighborhood of \( \infty \), \( Y(X) \sim X^{\sup\{k(n,p,m),1\}} \) if \( c > 0 \) or \( (k(n,p,m)-1)c \leq 0 \) and \( Y(X) \sim X^{n(p-1)} \) if \( c < 0 \) and \( k(n,p,m) < 1 \).

**Proof.** We consider the function \( H(x) = Mx^\theta \), where \( M \) and \( \theta \) are parameters to be determined. It is easy to see that \( H \) is a upper-solution of (2.2.10) (resp. lower-solution) if and only if, \( \forall s \in \mathbb{R}^* \),

\[
(2.2.30) \quad \theta M \geq cx^{1-\theta} + M^{-1/(p-1)}x^{p/(p-1)(k(s)-\theta)} + M^{-1/(p-1)} \left( \frac{f(x)}{x^q} - 1 \right) x^{p/(p-1)(k(s)-\theta)},
\]

respectively

\[
(2.2.31) \quad \theta M \leq cx^{1-\theta} + M^{-1/(p-1)}x^{p/(p-1)(k(s)-\theta)} + M^{-1/(p-1)} \left( \frac{f(x)}{x^q} - 1 \right) x^{p/(p-1)(k(s)-\theta)},
\]

We first examine the behavior at neighborhood of \( 0 \), we take \( s = q \) in (2.2.30) and (2.2.31) and using \( (h_1) \) we must have

\[
(2.2.32) \quad \frac{f(x)}{x^q} - 1 \to 0.
\]

We distinguish two cases:

**Case 1.** \( c > 0 \) or \( (k(q,p,m) - 1)c \geq 0 \). We obtain from (2.2.32): so that \( H \) satisfies (2.2.30) (resp. (2.2.31)) at neighborhood of \( 0 \) it is necessary that \( \inf\{Q(q),1\} \geq \theta \) (resp. \( \inf\{Q(q),1\} \leq \theta \)). We choose \( \theta = \inf\{Q(q),1\} \). Then we can have \( M_0 \) such that \( H \) is a upper-solution of (2.2.10) (resp. lower-solution) for all \( M > M_0 \) (resp. \( M < M_0 \)). Consequently \( y(x) \sim M_0x^{\inf\{Q(q),1\}} \).

**Case 2.** \( c < 0 \) and \( k(q,p,m) > 1 \). In this case, we choose \( \theta = m + q(p-1) \). Thus (2.2.30) (resp. (2.2.31)) becomes

\[
(2.2.33) \quad \theta M \geq \left[ c + M^{-1/(p-1)} + M^{-1/(p-1)} \left( \frac{f(x)}{x^q} - 1 \right) \right] x^{1-q(p-1)-m},
\]

respectively

\[
(2.2.34) \quad \theta M \leq \left[ c + M^{-1/(p-1)} + M^{-1/(p-1)} \left( \frac{f(x)}{x^q} - 1 \right) \right] x^{1-q(p-1)-m}.
\]

Since \( k(q,p,m) > 1 \) then \( 1 - q(p-1) - m < 0 \), and according to (2.2.32) we have (2.2.33) (resp. (2.2.34)) is checked for all \( M > (-c)^{1-p} \) (resp. \( M < (-c)^{1-p} \)) and consequently \( y(x) \sim (-c)^{1-p} x^{q(p-1)+m} \). By using the same method we can have the behavior at neighborhood of \( \infty \).

Now We consider \( \phi \) a solution of problem (2.2.28) and its maximum interval of existence is \( [-\infty, \beta] \). For \( \phi(\xi) \neq 0 \) (and consequently \( y(\phi(\xi)) \neq 0 \)) we have

\[
(2.2.35) \quad (p-1)/(m+p-1)y^{-1/(p-1)}(\phi^{(p-1)/(m+p-1)}(\xi)) \phi'(\xi) = 1.
\]
Integrating (2.2.26) on \((\xi, \xi_1) \subset \mathbb{R}, \beta\) we obtain

\[
(2.2.36) \quad \xi_1 - \xi = \int_\phi(\xi) y(g(s))^{-1/(p-1)} ds, \quad \forall \xi \in \mathbb{R}, \beta\text{ such that } \phi(\xi) > 0,
\]

with \(g(s) = s^{(p-1)/(m+p-1)}\).

If \(\phi\) never vanished on \([-\infty, \beta]\), we can tend \(\xi\) to \(-\infty\) in formula (2.2.36) to obtain \(\int_0^{\phi(\xi)} y^{-1/(p-1)}(g(s)) ds = \infty\). Then \(\phi\) is vanished if and only if

\[
(2.2.37) \quad \int_0^\infty y^{-1/(p-1)}(s) ds < \infty.
\]

In addition, we can tend \(\xi_1\) to \(\beta\) in the formula (2.2.36) to obtain \(\beta = \infty\) if and only if

\[
(2.2.38) \quad \int_0^{+\infty} y^{-1/(p-1)}(g(s)) ds = \infty.
\]

Thus, the proofs of Theorems 1.1 and 1.2 rises immediately by using the asymptotic behavior of \(y\) (solution of (2.2.10)). Moreover, we combine the relations (2.2.35) and the solution \(y\) in order to obtain the results of asymptotic behavior (Theorems 1.3 and 1.4).

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**References**


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