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Positive Solutions for Fourth Order Boundary Value Problems

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Abstract

In this paper, we investigate the problem of existence and nonexistence of positive solutions for the nonlinear boundary value problem:

$$u^{(4)}(t)+\lambda\, a(t)\, f(u(t))=0, \qquad 0\le t\le 1,$$

$$u(0)=u''(0)=u'''(0)=0, \quad \alpha\, u'(1)+\beta\, u''(1)=0.$$

By using Krasnoselskii's fixed-point theorem of cone, we establish various results on the existence of positive solutions of the boundary value problem. An example is also given to illustrate the main results.

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1-Introduction

One of the most frequently used tools for proving the existence of positive solutions to the integral equations and boundary value problems is Krasnoselskii's theorem on cone expansion and compression and its norm-type version due to Guo [3]. To the best of our knowledge, the first paper taking this approach is by Wang in [5]. Since this pioneering work, a lot research has been done in this area

[1, 2, 4, 6, 7, 8]. The purpose of this paper is to establish the existence of positive solutions to nonlinear fourth order boundary value problem:

$$u^{(4)}(t)+\lambda\, a(t)\, f(u(t))=0, \qquad 0\le t\le 1, \tag{1}$$

$$u(0)=u''(0)=u'''(0)=0, \quad \alpha u'(1)+\beta u''(1)=0. \tag{2}$$

where $\lambda>0$ is positive parameter, $a:(0,1)\to[0,\infty)$ is continuous and $\int_0^1 a(t)\,dt>0$, $f:[0,\infty)\to[0,\infty)$ is continuous and $\beta>\alpha$ Here, by a positive solution of the boundary value problem we mean a function which is positive on (0, 1) and satisfies differential equation (1) and the boundary condition (2).

2- Preliminaries

In this section, we present some notations and lemmas that will be used in the proof our main results.

Definition 1. Let E be a real Banach space. A nonempty closed convex set $K\subset E$ is called a ***cone*** of E if it satisfies the following conditions:

(1) $x\in K,\ \lambda\ge 0$ implies $\lambda x\in K$;

(2) $x\in K, -x\in K$ implies $x=0$.

Definition 2. An operator is called ***completely continuous*** if it is continuous and maps bounded sets into precompact sets**.**

Lemma 1. Let E be a Banach space and $K\subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0\in\Omega_1$ and $\overline{\Omega}_1\subset\Omega_2$. Let $T:K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ be a completely continuous operator. In addition suppose either

(H_1) $\|Tu\|\le\|u\|, \forall u\in K\cap\partial\Omega_1$ and $\|Tu\|\ge\|u\|, \forall u\in K\cap\partial\Omega_2$ or

(H_2) $\|Tu\|\le\|u\|, \forall u\in K\cap\partial\Omega_2$ and $\|Tu\|\ge\|u\|, \forall u\in K\cap\partial\Omega_1$,

holds. Then T has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Lemma 2. Let $y\in C[0,1]$, then the boundary value problem

$$u^{(4)}(t)+y(t)=0, \qquad 0\le t\le 1, \tag{3}$$

$$u(0)=u''(0)=u'''(0)=0, \quad \alpha u'(1)+\beta u''(1)=0. \tag{4}$$

has a unique solution $u(t)=\int_0^1 G(t,s)y(s)\,ds,$

where

$$G(t,s)=\begin{cases}\dfrac{\beta}{\alpha}t(1-s)+\dfrac{t(1-s)^2}{2!} & 0\le t\le s\le 1,\\[2mm] \dfrac{\beta}{\alpha}t(1-s)+\dfrac{t}{2!}(1-s)^2-\dfrac{(t-s)^3}{3!}, & 0\le s\le t\le 1.\end{cases}$$

Proof: Applying the Laplace transform to Eq(3) we get

$$s^4 u(s) - s^3 u(0) - s^2 u'(0) - s\, u''(0) - u'''(0) = -y(t), \qquad (5)$$

The Laplace inversion of Eq. (5) gives the final solution as

$$u(t) = \frac{\beta}{\alpha} t \int_0^1 (1-s)\, y(s)\, ds + t \int_0^1 \frac{(1-s)^2}{2!} y(s) ds - \int_0^t \frac{(t-s)^3}{3!} y(s) ds \qquad (6)$$

The proof is complete.

It is obvious that

$$G(t,s) \geq 0 \quad \text{and} \quad G(t,s) \leq G(1,s), \qquad 0 \leq t,s \leq 1. \qquad (7)$$

Lemma 3. $G(t,s) \geq q(t) G(1,s)$ *for* $0 \leq t,s \leq 1$, where $q(t) = t$

Proof: If $t \geq s$,then

$$\frac{G(t,s)}{G(1,s)} = \frac{6\beta t(1-s) + 3\alpha t(1-s)^2 - \alpha(t-s)^3}{6\beta(1-s) + 3\alpha(1-s)^2 - \alpha(1-s)^3}$$

$$\geq \frac{6\beta t(1-s) + 3\alpha t(1-s)^2 - \alpha(t-s)(1-s)^2}{6\beta(1-s) + 3\alpha(1-s)^2 - \alpha(1-s)^3} \geq t.$$

If $t \leq s$,then $\frac{G(t,s)}{G(1,s)} = t$. The proof is complete.

3. Main results

In this section, we will apply Krasnoselskii's fixed point theorem to the eigenvalue problem (1),(2). We note that u(t) is a solution of(1),(2) if and only if

$$u(t) = \lambda \int_0^1 G(t,s) a(s) f(u(s))\, ds, \qquad 0 \leq t \leq 1.$$

For our constructions, we shall consider the Banach space $X = C[0,1]$ equipped with standard norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|, u \in X$. Define a cone P by:

$$P = \{u \in X :\ u(t)\ ,\ u(t) \geq q(t)\|u\|,\ t \in [0,1]\}\ .$$

It is easy to see that if $u \in P$, then $\|u\| = u(1)$. Define an integral operator by

$$T\,u(t) = \lambda \int_0^1 G(t,s)\, a(s)\, f(u(s))\, ds, \qquad 0 \leq t \leq 1, \quad u \in P. \qquad (8)$$

Lemma 4. $T(P) \subset P.$

Proof. Notice from (5) and (7) that, for $u \in P$, $T\,u(t) \geq 0$ on [0, 1]and

$$T\,u(t) = \lambda \int_0^1 G(t,s) a(s) f(u(s))\, ds \geq \lambda\, q(t) \int_0^1 G(1,s) a(s) f(u(s))\, ds$$

$$\geq \lambda\, q(t) \max_{0 \leq t \leq 1} \int_0^1 G(t,s) a(s) f(u(s))\, ds = q(t)\ \|T\,u\|.$$

Thus $T(P) \subset P.$ By standard argument, it is easy to see that $T : P \to P$ is a completely continuous operator.

Following Sun and Wen [7], we define some important constants

$$A=\int_0^1 G(1,s)\,a(s)\,q(s)\,\mathrm{ds}, \qquad B=\int_0^1 G(1,s)\,a(s)\,\mathrm{ds},$$

$$F_0=\limsup_{u\to 0^+}\frac{f(u)}{u}, \qquad f_0=\liminf_{u\to 0^+}\frac{f(u)}{u}.$$

$$F_\infty=\limsup_{u\to+\infty}\frac{f(u)}{u}, \qquad f_\infty=\liminf_{u\to+\infty}\frac{f(u)}{u}.$$

Here we assume that $\frac{1}{A f_\infty}=0$ if $f_\infty=\infty$ and $\frac{1}{B F_0}=\infty$ if $F_0=0$ and $\frac{1}{A f_0}=0$ if $f_0=\infty$ and $\frac{1}{B F_\infty}=\infty$ if $F_\infty=0$.

In this section, we will apply Krasnoselskii's fixed-point theorem to the eigenvalue problem (1),(2).

Theorem 1. Suppose that $A f_\infty > B F_0$. Then for each $\lambda\in(\frac{1}{A f_\infty},\frac{1}{B F_0})$ the problem (1),(2) has at least one positive solution.

Proof: we choose $\varepsilon>0$ sufficiently small such that $(F_0+\varepsilon)\lambda B\le 1$. By the definition of F_0, we can see that there exists $l_1>0$, such that $f(u)\le(F_0+\varepsilon)u$ for $0<u\le l_1$. If $u\in P$ with $\|u\|=l_1$, we have

$$\|Tu\|=(Tu)(1)=\lambda\int_0^1 G(1,s)\,a(s)\,f(u(s))\,ds$$

$$\le\lambda\int_0^1 G(1,s)\,a(s)\,(F_0+\varepsilon)u(s)\le\lambda(F_0+\varepsilon)\|u\|B\le\|u\|.$$

Then we have $\|Tu\|\le\|u\|$. Thus if we let $\Omega_1=\{u\in X\,\big|\,\|u\|<l_1\}$, then $\|Tu\|\le\|u\|$ for $u\in P\cap\partial\Omega_1$.

Following Yang [8], we choose $\delta>0$ and $c\in(0,\frac{1}{4})$, such that $\lambda\left((f_\infty-\delta)\int_c^1 G(1,s)\,a(s)\,q(s)\,ds\right)\ge 1$. There exists $l_3>0$, such that $f(u)\ge(f_\infty-\delta)u$ for $u>l_3$. Let $l_2=\max\left\{\frac{l_3}{q(c)},2l_1\right\}$. If $u\in P$ with $\|u\|=l_2$, then for each $t\in[c,1]$, we have $u(t)\ge q(t)l_2\ge q(c)l_2\ge l_3$.

Therefore, for each $u\in P$ with $\|u\|=l_2$, we have

$$\|Tu\|=(Tu)(1)=\lambda\int_0^1 G(1,s)\,a(s)\,f(u(s))\,ds\ge\lambda\int_c^1 G(1,s)\,a(s)\,(f_\infty-\delta)u(s)\,ds$$

$$\ge\lambda(f_\infty-\delta)\|u\|\int_c^1 G(1,s)\,a(s)\,q(s)\,ds\ge\|u\|.$$

Thus if we let $\Omega_2 = \{u \in X \mid \|u\| < l_2\}$, then $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. Condition ($H_1$) of Krasnoselskii's fixed-point theorem is satisfied. So there exists a fixed point of T in P. This completes the proof.

Theorem 2. Suppose that $A f_0 > B F_\infty$. Then for each $\lambda \in (\frac{1}{A f_0}, \frac{1}{B F_\infty})$ the problem (1),(2) has at least one positive solution.

The proof of Theorem 2 is very similar to that of Theorem 1 and therefore omitted.

Theorem 3. Suppose that $\lambda B f(u) < u$ for $u \in (0,\infty)$. Then the problem (1),(2) has no positive solution.

Proof: Following Sun and Wen [7], assume to the contrary that u is a positive solution of (1),(2). Then

$$u(1) = \lambda \int_0^1 G(1,s)\,a(s)\,f(u(s))\,ds < \frac{1}{B}\int_0^1 G(1,s)\,a(s)\,u(s)\,ds \leq \frac{u(1)}{B}\int_0^1 G(1,s)\,a(s)\,ds = u(1)$$

This is a contradiction and completes the proof.

Theorem 4. Suppose that $\lambda A f(u) > u$ for $u \in (0,\infty)$. Then the problem (1),(2) has no positive solution.

Proof: Assume to the contrary that u is a positive solution of (1),(2)Then

$$u(1) = \lambda \int_0^1 G(1,s)\,a(s)\,f(u(s))\,ds \geq \frac{\|u\|}{A}\int_0^1 G(1,s)\,a(s)\,q(s)\,ds \geq u(1).$$

This is a contradiction and completes the proof.

Example 1: Consider the boundary value problem

$$u^{(4)}(t) + \lambda\ (10s^2 + 2)\frac{7u^2 + u}{u+1}(8 + \sin u) = 0, \qquad 0 \leq t \leq 1, \tag{9}$$

$$u(0) = u'(0) = u''(0) = 0,\ \ 2u'(1) + 5u''(1) = 0. \tag{10}$$

Then $F_0 = f_0 = 8,\ F_\infty = 63,\ f_\infty = 49$, and $8u < f(u) < 63u$. By direct calculations, we obtain that $A = 2.22143$ and $B = 4.97222$. From theorem 1 we see that if $\lambda \in (0.00918695, 0.0251397)$ then the problem (9) and (10) has a positive solution. From theorem (3) we have that if $\lambda < 0.00319234$ then the problem (9) and (10) has no positive solution. By theorem (4), if $\lambda > 0.0562701$ then the problem (9) and (10) has no positive solution.

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