Coefficient Bounds for Certain Classes
of Close-to-Convex Functions

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Abstract

Let $R$ denote the subclass of normalized analytic univalent functions $f$ defined
by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and satisfy
$$\text{Re} \left[ e^{i\alpha} f'(z) \right] > 0$$
where $z \in D = \{z : |z| < 1\}$. The object of the present paper is to introduce the
functional $|a_2a_4 - \mu a_3^2|$. For $f \in R$, we give sharp upper bound for $|a_2a_4 - \mu a_3^2|$.

Mathematics Subject Classification: Primary 30C45

Keywords: Univalent function, close-to-convex, Hankel determinant

1 Introduction and definitions

Let $A$ denote the class of functions of the form
which are analytic in the open unit disk $D = \{ z \in C : |z| < 1 \}$. Also let $S$, $S^\ast (\beta)$ and $K(\beta)$ denote the subclasses of $A$ consisting of functions which are univalent, starlike of order $\beta$ and convex of order $\beta$ in $D$. In particular, the classes $S^\ast (0) = S^\ast$ and $K(0) = K$ are the familiar ones of starlike and convex functions in $D$, respectively.

Janteng et al. [1] had established on the second Hankel determinant for functions $f \in S$ and $K$. In this paper, we will follow the same procedure or method produce by them in finding $|a_2a_4 - \mu a_3^2|$ but for certain classes of close-to-convex functions.

**Definition 1.1.** Let $f$ be given by (1.1). Then $f \in R$ if it satisfies the inequality
\[
\Re \left( e^{i\alpha} f'(z) \right) > 0, \quad z \in D.
\] (1.2)

The subclass $R$ was studied systematically by Silverman and Silvia [4].

**Definition 1.2.** Let $f$ be given by (1.1). Then $f \in S^\ast$ if and only if
\[
\Re \left( \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in D.
\] (1.3)

**Definition 1.3.** Let $f$ be given by (1.1). Then $f \in K$ if and only if
\[
\Re \left( \frac{(zf'(z))'}{f'(z)} \right) > 0, \quad z \in D.
\] (1.4)

It follows that $f \in K$ if and only if $zf'(z) \in S^\ast$.

**Definition 1.4.** Let $f$ be given by (1.1). Then $f$ is called close-to-convex in $D$ if, and only if, for $z \in D$, and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ there exists $\phi \in S^\ast$ such that
\[
\Re \left( e^{i\alpha} \frac{zf''(z)}{\phi(z)} \right) > 0.
\] (1.5)

We shall denote the class of all functions by $C$.

By Alexander’s Theorem if $h(z)$ is convex, then $\phi(z) = zh'(z)$ is starlike. Hence in Definition 1.4 we can replace (1.5) with

**Definition 1.5.** Let $f$ be given by (1.1). Then $f$ is called close-to-convex in $D$ if, and only if, for $z \in D$, and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ there exists $h \in K$ such that
\[
\Re \left( e^{i\alpha} \frac{f'(z)}{h'(z)} \right) > 0.
\] (1.6)

Silverman and Silvia [4] define the function to be $\alpha$-close-to-convex function.
2 Lemmas

The following lemmas will be required in our investigation.

Let $P$ be the family of all functions $p$ analytic in $D$ for which $\Re \{ p(z) \} > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

for $z \in D$.

**Lemma 2.1** ([3]). If $p \in P$ then $|c_k| \leq 2$ for each $k$.

**Lemma 2.2** ([7]). The power series for $p(z)$ given by (2.1) converges in $D$ to a function in $P$ if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots$$

and $c_{-k} = \overline{c_k}$, are all nonnegative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{i \theta_k} z) \quad \rho_k > 0, \ t_k \text{ real and } t_k \neq t_j \text{ for } k \neq j; \text{ in this case }$$

$$D_n > 0 \text{ for } n < m-1 \text{ and } D_n = 0 \text{ for } n \geq m.$$

**Lemma 2.3.** (cf. [5, page 254], see also [6]). Let the function $p \in P$ be given by the power series (2.1). Then,

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

for some $x$, $|x| \leq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2) c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)$$

for some $z$, $|z| \leq 1$.

**Lemma 2.4.** Let $f \in S^*$. Then

$$a_3^2 - \frac{8a_3^3}{9} \leq 1.$$

The result obtained is sharp.

**Proof.**

Since $\frac{zf''(z)}{f(z)} \in C$, it follows from (1.3) that $\exists p \in P$ such that

$$zf''(z) = f(z)p(z)$$

for some $z \in D$. Equating coefficients in (2.5) yields
\( a_2 = c_1 \)
\( a_3 = \frac{c_2^2 + c_1^2}{2} \)
\( a_4 = \frac{c_2^3 + c_1c_2 + c_1^3}{6} \)

From (2.6), it is easily established that
\[
\left| a_2 a_4 - \frac{8a_3^2}{9} \right| = \left| \frac{c_3 c_1}{3} - \frac{2c_2^2}{9} + \frac{c_2 c_1}{18} - \frac{c_1}{18} \right|.
\]

Now, assuming \( c_1 = c(0 \leq c \leq 2) \) and using (2.3) together with (2.4) we have
\[
\left| \frac{c_3 c_1}{3} - \frac{2c_2^2}{9} + \frac{c_2 c_1}{18} - \frac{c_1}{18} \right| = \frac{c^4}{6} + \frac{5c^2(4-c^2)x}{12} - \frac{(7c^2 + 8)(4-c^2)x^2}{36} + \frac{c(4-c^2)(1-|x|^2)x^2}{2}.
\]

Application of the triangle inequality gives
\[
\left| \frac{c_3 c_1}{3} - \frac{2c_2^2}{9} + \frac{c_2 c_1}{18} - \frac{c_1}{18} \right| \leq \frac{c^4}{6} + \frac{c(4-c^2)}{2} + \frac{5c^2(4-c^2)x}{12} + \frac{(7c^2 - 4)(4-c^2)x}{36} + \frac{c(4-c^2)(1-|x|^2)x^2}{2} = F(\rho)
\]

with \( \rho = |x| \leq 1 \). For
\[
F'(\rho) = \frac{5c^2(4-c^2)}{12} + \frac{(7c^2 - 4)(4-c^2)x}{36}
\]

it can be shown that \( F'(\rho) > 0 \) and thus is an increasing function implying \( \text{Max}_{\rho \leq 1} F(\rho) = F(1) \). Now let
\[
G(c) = F(1) = \frac{c^4}{6} + \frac{c(4-c^2)}{2} + \frac{5c^2(4-c^2)}{12} + \frac{(7c^2 - 4)(4-c^2)}{36}.
\]

Trivially, one can show that \( G \) has a maximum attained at \( c = 1 \). The upper bound for (2.8) corresponds to \( \rho = 1 \) and \( c = 1 \), in which case
\[
\left| \frac{c_3 c_1}{3} - \frac{2c_2^2}{9} + \frac{c_2 c_1}{18} - \frac{c_1}{18} \right| \leq 1.
\]

Letting \( c_1 = 1 \), \( c_2 = -1 \) and \( c_3 = -2 \) in (2.8) shows that the result is 1 and sharp.

This completes the proof of Lemma 2.4.

**Lemma 2.5.** Let \( f \in K \). Then
The result obtained is sharp.

The proof of the lemma can be found in Janteng et al. [1].

3 Results

Theorem 3.1. Let \( f \in R \). Then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{8}.
\]

The result obtained is sharp.

Proof.
We refer to the method by Libera and Zlotkiewicz [5,6]. Since \( f \in R \), it follows from (1.2) that

\[
e^{ia} f'(z) = p(z)
\]

for some \( z \in D \). Equating coefficients in (2.3) yields

\[
\begin{aligned}
2a_2e^{ia} &= c_1 \\
3a_3e^{ia} &= c_2 \\
4a_4e^{ia} &= c_3
\end{aligned}
\]

(3.2)

From (3.2), it can be easily established that

\[
|a_2a_4 - a_3^2| = \left| \frac{c_1c_3 - c_2^2}{8} - \frac{c_2}{9} \right|.
\]

and the rest will follow on the proof given by Janteng et al. [2], in which case by letting \( c_1 = 1, c_2 = -1 \) and \( c_3 = -2 \) we have

\[
\left| \frac{c_1c_3 - c_2^2}{8} - \frac{c_2}{9} \right| \leq \frac{13}{36}.
\]

This concludes the proof of our theorem.

Theorem 3.2. Let \( f \in R, \phi \in S^* \) and \( \frac{zf'(z)}{\phi(z)} \in C \). Then

\[
|a_2a_4 - a_3^2| \leq \frac{71}{72}.
\]

The result obtained is sharp.

Proof.
Since \( \frac{zf'(z)}{\phi(z)} \in C \), it follows from (1.5) that \( \exists p \in P \) such that

\[
zf'(z) = \phi(z)p(z)
\]

(3.3)

for some \( z \in D \). Equating coefficients in (3.3) yields
Also, since $\phi(z) \in S^*$, it follows from (1.3) that $\exists p \in P$ such that
definition of $z\phi'(z) = \phi(z)p(z)$}
for some $z \in D$. Equating coefficients in (3.5) yields

gives

From (3.4) and (3.6), it is easily established that

From Lemma 2.4, $\frac{\phi_{2\phi_4}}{8} - \frac{\phi_3^2}{9} \leq \frac{1}{8}$.

Now, assuming $c_1 = c(0 \leq c \leq 2)$ and using (2.3) together with (3.3) we have

Application of the triangle inequality gives

with $\rho = |x| \leq 1$. For

it can be shown that $F'(\rho) > 0$ and thus is an increasing function implying
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Max_{\rho \leq 1} F(\rho) = F(1). Now let

G(c) = F(1) = \frac{c^4}{12} + \frac{c}{9} + \frac{7c(4-c^2)}{48} + \frac{7c^2(4-c^2)}{288} + \frac{(4-c^2)(7c-2)(c-2)}{96}.

Trivially, one can show that G has a maximum attained at \( c = 1 \). The upper bound for (3.8) corresponds to \( \rho = 1 \) and \( c = 1 \), in which case

\[
\left| \frac{7c_3c_1}{24} - \frac{11(c_1^2c_2 + c_1^4)}{144} - \frac{c_2^2}{6} - \frac{c_1}{9} \right| \leq \frac{31}{36}.
\]

Letting \( c_1 = 1, \ c_2 = -1 \) and \( c_3 = -2 \) in (3.7) shows that the result is \( \frac{71}{72} \) and sharp. This completes the proof of Theorem 3.2.

**Theorem 3.3.** Let \( f \in R, \ h \in K \) and \( \frac{f'(z)}{h'(z)} \in C \). Then

\[
\left| a_2a_4 - a_3^2 \right| \leq 1.
\]

The result obtained is sharp.

**Proof.**

Since \( \frac{f'(z)}{h'(z)} \in C \), it follows from (1.6) that \( \exists p \in P \) such that

\[
f''(z) = h'(z)p(z)
\]

for some \( z \in D \). Equating coefficients in (3.9) yields

\[
\begin{align*}
2a_2 &= 2d_2 + c_1 \\
3a_3 &= 3d_3 + 2d_2c_1 + c_2 \\
4a_4 &= 4d_4 + 3d_3c_1 + 2d_2c_2 + c_3
\end{align*}
\]

(3.10)

Also, since \( h'(z) \in K \), it follows from (1.4) that \( \exists p \in P \) such that

\[
(zh'(z))' = h'(z)p(z)
\]

(3.11)

for some \( z \in D \). Equating coefficients in (3.11) yields

\[
\begin{align*}
d_2 &= \frac{c_1}{2} \\
d_3 &= \frac{c_2 + c_1^2}{6} \\
d_4 &= \frac{c_3 + c_1c_2 + c_1^3}{12} + \frac{c_1^2}{8} + \frac{c_1^3}{24}
\end{align*}
\]

(3.12)

From (3.10) and (3.12), it is easily established that

\[
\left| a_2a_4 - a_3^2 \right| \leq \left| d_2d_4 - d_3^2 \right| + \frac{5c_1^4}{144} + \frac{7c_3c_1}{24} - \frac{c_2^2}{144} - \frac{c_2}{9} - \frac{c_1^3}{9}.
\]

(3.13)

From Lemma 2.5, \( \left| d_2d_4 - d_3^2 \right| \leq \frac{1}{8} \).

Now, assuming \( c_1 = c (0 \leq c \leq 2) \) and using (2.3) together with (2.4) we have
Application of the triangle inequality gives
\[
\left| \frac{5c_1^4 + 7c_2c_1 - c_1^2c_2 - 2c_2^2 - c_1^3}{144} - \frac{c_1^2(4 - c_2^2)}{9} \right| \\
\leq \frac{(7c - 16)c_3^3}{144} + \frac{c_1^2(4 - c_2^2)}{32} \left( \frac{5c_2^2 + 64}{288} \right) x^2 + \frac{7c(4 - c_2^2)(1 - |x|^2)^2}{48}.
\]

With \( \rho = |x| \leq 1 \), one can show that
\[
F'(\rho) = \frac{c_1^2(4 - c_2^2)}{32} \left( \frac{5c_2^2 + 64}{288} \right) \rho
\]

it can be shown that \( F'(\rho) > 0 \) and thus is an increasing function implying
\[
Max\_{\rho \leq 1} F(\rho) = F(1).
\]

Letting \( 1_{12} = c, 1_{12} = c, \) and \( 2_{12} = c \) in (3.13) shows that the result is 1 and sharp. This completes the proof of Theorem 3.3.

References


Received: June 10, 2008