A Note on Fixed Point and Common Fixed Point Theorems in 2-Metric Spaces

Aage C. T.

Department of Mathematics
North Maharashtra University, Jalgaon
cage17@gmail.com

Salunke J. N.

Department of Mathematics
North Maharashtra University, Jalgaon
drjnsalunke@gmail.com

Abstract

This paper presents some fixed point theorems and common fixed point theorems in 2-metric spaces.

Mathematics Subject Classification: 54H25, 47H10

Keywords: fixed point, 2-metric space

1 Introduction

The concept of 2-metric space was initiated by S. Gahler [9]. The study was further enhanced by B. E. Rhoades[4], K. Iseki[5], A. K. Sharma[1, 2, 3], M. S. Khan[7] and M. Ashraf[6]. Moreover B. E. Rhoades and others introduced several properties of 2-metric spaces and proved some fixed point theorems for contractive and expansion mappings. In this same way, we prove a fixed point theorem and common fixed point theorems for the mappings satisfying different types of contractive conditions in 2-metric space.

2 Preliminary Notes

Definition 2.1 A 2-metric space is a space \( X \) in which for each triple of points, \( x, y, z \), there exists a real function \( d(x, y, z) \) such that
(i) To each pair of distinct points \( x, y \) in \( X \), there exists a point \( z \in X \) such that \( d(x, y, z) \neq 0 \),

(ii) \( d(x, y, z) = 0 \), when at least two of \( x, y, z \) are equal,

(iii) \( d(x, y, z) = d(y, z, x) = d(x, z, y) \),

(iv) \( d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z) \), for all \( w \in X \).

From the above definition it is clear that \( d(x, y, z) \) is nonnegative.

**Definition 2.2** A sequence in a 2-metric space \( (X, d) \) is said to be convergent with limit \( x \in X \) if \( \lim d(x_n, x, a) = 0 \), for all \( a \in X \). It follows that if the sequence \( \{x_n\} \) converges to \( x \) then \( \lim_{n \to \infty} d(x_n, a, b) = d(x, a, b) \) for all \( a, b \in X \).

**Definition 2.3** A sequence in a 2-metric space \( X \) is Cauchy if \( \lim_{n,m \to \infty} d(x_m, x_n, a) = 0 \), for all \( a \in X \).

**Proposition 2.4** If a sequence is convergent in a 2-metric space then it is a Cauchy sequence.

**Proposition 2.5** Limit of a sequence in a 2-metric space, if exist, is unique.

**Proposition 2.6** If a sequence \( \{x_n\} \) in a 2-metric space converges to \( x \) then every subsequence of \( \{x_n\} \) also converges to the same limit \( x \).

**Definition 2.7** A 2-metric space \( (X, d) \) is said to be complete if every Cauchy sequence in \( X \) is convergent.

### 3 Main Results

**Theorem 3.1** Let \( (X, d) \) be a complete 2-metric Space. Let \( E \) be a continuous self map of \( X \), satisfying the conditions:

\[
(i) \quad d^2(Ex, Ey, a) \leq \alpha d(x, Ex, a) d(y, Ey, a) + \beta d(x, E, a) d(y, Ex, a) + \gamma d(y, Ey, a) d(y, Ex, a) + \delta d(x, Ey, a) d(y, Ex, a),
\]

for all \( x, y, a \in X \) and \( \alpha, \beta, \gamma, \delta \geq 0 \) with \( \max \{\alpha, \delta\} < 1 \). Then \( E \) has a unique fixed point in \( X \).

**Proof:**

Let \( x_0 \) be an arbitrary point in \( X \); define sequence \( \{x_n\} \) recurrently, \( Ex_0 = x_1, Ex_1 = x_2, \cdots Ex_n = x_{n+1}, \) where, \( n = 0, 1, 2, 3, \cdots \). Now by (i) we have

\[
d^2(x_1, x_2, a) = d^2(Ex_0, Ex_1, a)
\]
\[
\begin{align*}
&\leq \alpha d(x_0, Ex_0, a)d(x_1, Ex_1, a) + \beta d(x_0, Ex_0, a)d(x_1, Ex_0, a) \\
&\quad + \gamma d(x_1, Ex_1, a)d(x_1, Ex_0, a) + \delta d(x_0, Ex_1, a)d(x_1, Ex_0, a) \\
&= \alpha d(x_0, x_1, a)d(x_1, x_2, a) + \beta d(x_0, x_1, a)d(x_1, x_1, a) \\
&\quad + \gamma d(x_1, x_2, a)d(x_1, x_1, a) + \delta d(x_0, x_2, a)d(x_1, x_1, a)
\end{align*}
\]

i.e.
\[
\begin{align*}
d^2(x_1, x_2, a) &\leq \alpha d(x_0, x_1, a)d(x_1, x_2, a) \\
d(x_1, x_2, a) &\leq \alpha d(x_0, x_1, a)
\end{align*}
\]

Similarly,
\[
\begin{align*}
d(x_2, x_3, a) &\leq \alpha d(x_1, x_2, a) \\
&\leq \alpha \alpha d(x_0, x_1, a) \\
&= \alpha^2 d(x_0, x_1, a) \\
&\quad \cdots \\
&\quad \cdots
\end{align*}
\]

i.e.
\[
\begin{align*}
d(x_n, x_{n+1}, a) &\leq \alpha^n d(x_0, x_1, a).
\end{align*}
\]

We claim that the sequence \(\{x_n\}\) is a Cauchy sequence in \(X\).

For \(m > n\), we have
\[
\begin{align*}
d(x_n, x_m, a) &\leq d(x_n, x_m, x_{n+1}) + d(x_n, x_{n+1}, a) + d(x_{n+1}, x_m, a) \\
&= d(x_n, x_{n+1}, x_m) + d(x_n, x_{n+1}, a) + d(x_{n+1}, x_m, a) \\
&\leq \alpha^n d(x_0, x_1, x_m) + \alpha^n d(x_0, x_1, a) + d(x_{n+1}, x_m, a) \\
&\leq \alpha^n d(x_0, x_1, x_m) + \alpha^n d(x_0, x_1, a) + d(x_{n+1}, x_m, x_{n+2}) \\
&\quad + d(x_{n+1}, x_{n+2}, a) + d(x_{n+2}, x_m, a) \\
&\leq (\alpha^n + \alpha^{n+1})d(x_0, x_1, x_m) \\
&\quad + (\alpha^n + \alpha^{n+1})d(x_0, x_1, a) + d(x_{n+2}, x_m, a) \\
&\quad \cdots \\
&\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-2})d(x_0, x_1, x_m) \\
&\quad + (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-2})d(x_0, x_1, a) + d(x_{m-1}, x_m, a) \\
&\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-2})d(x_0, x_1, x_m) \\
&\quad + (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-2} + \alpha^{m-1})d(x_0, x_1, a) \\
&\leq \frac{\alpha^n}{1 - \alpha}d(x_0, x_1, x_m) + \frac{\alpha^n}{1 - \alpha}d(x_0, x_1, a).
\end{align*}
\] (1)
Again, we have
\[
d(x_0, x_1, x_m) \leq d(x_0, x_1, x_{m-1}) + d(x_0, x_{m-1}, x_m) + d(x_{m-1}, x_1, x_m)
\]
\[
= d(x_0, x_1, x_{m-1}) + d(x_{m-1}, x_m, x_0) + d(x_{m-1}, x_m, x_1)
\]
\[
\leq d(x_0, x_1, x_{m-1}) + \alpha^{m-1}d(x_0, x_1, x_0) + \alpha^{m-2}d(x_1, x_2, x_1)
\]
\[
= d(x_0, x_1, x_{m-1})
\]
i.e.
\[
d(x_0, x_1, x_m) \leq d(x_0, x_1, x_{m-1})
\]
\[
\leq d(x_0, x_1, x_{m-2})
\]
\[
\cdots
given that if
\[
d(x_0, x_1, x_m) \leq d(x_0, x_1, x_1) = 0.
\]
Hence \(d(x_0, x_1, x_m) = 0\).

As \(n, m \to \infty\) in eq (1), we have
\[
d(x_n, x_m, a) \to 0.
\]

Thus \(\{x_n\}\) is a Cauchy sequence in the complete 2-metric space, there exist a point \(u \in X\) such that \(\{x_n\} \to u\). Since \(E\) is continuous, by proposition (2.6) we have
\[
E(u) = \lim E(x_n) = \lim x_{n+1} = u.
\]

Thus \(E(u) = u\). So \(E\) has a fixed point.

**Uniqueness:**

In order to prove that \(u\) is the unique fixed point of \(E\). Consider \(u, v\) as fixed point of \(E\). Then \(d(u, v, a) = d(Eu, Ev, a)\) and
\[
d^2(u, v, a) = d^2(Eu, Ev, a)
\]
\[
\leq \alpha d(u, Eu, a)d(v, Ev, a) + \beta d(u, Eu, a)d(v, Eu, a)
\]
\[
+ \gamma d(v, Ev, a)d(v, Eu, a) + \delta d(u, Ev, a)d(v, Eu, a)
\]
\[
= \alpha d(u, u, a)d(v, v, a) + \beta d(u, u, a)d(v, v, a)
\]
\[
+ \gamma d(v, v, a)d(v, v, a) + \delta d(u, v, a)d(v, u, a)
\]
\[
= \delta d^2(u, v, a)
\]
i.e. \(d^2(u, v, a) \leq \delta d^2(u, v, a)\). This proves that \(d^2(u, v, a) = 0\) i.e. \(d(u, v, a) = 0\) for all \(a \in X\), since \(\delta < 1\). Therefore \(u = v\) and hence uniqueness of fixed point follows.
Theorem 3.2 Let \((X,d)\) be a complete 2-metric space. Let \(E\) and \(T\) be two continuous self mappings of \(X\), satisfying the conditions:

(i) \(ET = TE\), \(E(X) \subset T(X)\)

(ii) 
\[
d^2(Ex, Ey, a) \leq \alpha d(Tx, Ex, a)d(Ty, Ey, a) + \beta d(Tx, Ex, a)d(Ty, Ex, a) \\
+ \gamma d(Ty, Ey, a)d(Ty, Ex, a) + \delta d(Tx, Ey, a)d(Ty, Ex, a),
\]

for all \(x, y, a \in X\) and \(\alpha, \beta, \gamma \geq 0\) with \(\max \{\alpha, \delta\} < 1\). Then \(E\) and \(T\) have a unique common fixed point in \(X\).

Proof:
Let \(x_0\) be an arbitrary point in \(X\), since \(E(X) \subset T(X)\), we can choose \(x_1 \in X\) such that \(Ex_0 = Tx_1, Ex_1 = Tx_2, \ldots, Ex_n = Tx_{n+1}\). For \(n = 1, 2, 3, \ldots\) we have \(d(Tx_{n+1}, Tx_{n+2}, a) = d(Ex_n, Ex_{n+1}, a)\), for all \(a \in X\).

\[
d^2(Tx_{n+1}, Tx_{n+2}, a) \\
= d^2(Ex_n, Ex_{n+1}, a) \\
\leq \alpha d(Tx_n, Ex_n, a)d(Tx_{n+1}, Ex_{n+1}, a) + \beta d(Tx_n, Ex_n, a)d(Tx_{n+1}, Ex_n, a) \\
+ \gamma d(Tx_{n+1}, Ex_{n+1}, a)d(Tx_{n+1}, Ex_n, a) + \delta d(Tx_n, Ex_{n+1}, a)d(Tx_{n+1}, Ex_n, a) \\
= \alpha d(Tx_n, Tx_{n+1}, a)d(Tx_{n+1}, Tx_{n+2}, a) + \beta d(Tx_n, Tx_{n+1})d(Tx_{n+1}, Tx_{n+2}, a) \\
+ \gamma d(Tx_{n+1}, Tx_{n+2}, a)d(Tx_{n+1}, Tx_{n+1}, a) + \delta d(Tx_n, Tx_{n+2}, a)d(Tx_{n+1}, Tx_{n+1}, a) \\
= \alpha d(Tx_n, Tx_{n+1}, a)d(Tx_{n+1}, Tx_{n+2}, a),
\]

\[
\Rightarrow \, d(Tx_{n+1}, Tx_{n+2}, a) \leq \alpha d(Tx_n, Tx_{n+1}, a).
\]

Also \(d(Tx_n, Tx_{n+1}, a) \leq \alpha d(Tx_{n-1}, Tx_n, a)\) for all \(a \in X\). Hence

\[
d(Tx_{n+1}, Tx_{n+2}, a) \leq \alpha^n d(Tx_1, Tx_2, a) \\
\leq \alpha^{n+1} d(Tx_0, Tx_1, a).
\]

We claim that the sequence \(\{Tx_n\}\) is a Cauchy sequence in \(X\).
For \(m > n\), we have

\[
d(Tx_n, Tx_m, a) \leq d(Tx_n, Tx_m, Tx_{n+1}) + d(Tx_n, Tx_{n+1}, a) + d(Tx_{n+1}, Tx_m, a) \\
= d(Tx_n, Tx_{n+1}, Tx_m) + d(Tx_n, Tx_{n+1}, a) + d(Tx_{n+1}, Tx_m, a) \\
\leq \alpha^n d(Tx_0, Tx_1, Tx_m) + \alpha^n d(Tx_0, Tx_1, a) + d(Tx_{n+1}, Tx_m, a) \\
\leq \alpha^n d(Tx_0, Tx_1, Tx_m) + \alpha^n d(Tx_0, Tx_1, a) \\
+ d(Tx_{n+1}, Tx_m, Tx_{n+2}) \\
+ d(Tx_{n+1}, Tx_{n+2}, a) + d(Tx_{n+2}, Tx_m, a) \\
\leq (\alpha^n + \alpha^{n+1})d(Tx_0, Tx_1, Tx_m)
\]
\begin{align*}
+ (\alpha^n + \alpha^{n+1})d(Tx_0, Tx_1, a) + d(Tx_{n+2}, Tx_m, a) \\
\cdots \\
\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-2})d(Tx_0, Tx_1, Tx_m) \\
+ (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-2})d(Tx_0, Tx_1, a) \\
+ d(Tx_{m-1}, Tx_m, a) \\
\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-2})d(Tx_0, Tx_1, Tx_m) \\
+ (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-2} + \alpha^{m-1})d(Tx_0, Tx_1, a) \\
\leq \frac{\alpha^n}{1 - \alpha}d(Tx_0, Tx_1, Tx_m) + \frac{\alpha^n}{1 - \alpha}d(Tx_0, Tx_1, a). 
\end{align*}

Again, we have
\begin{align*}
    d(Tx_0, Tx_1, Tx_m) &\leq d(Tx_0, Tx_1, Tx_{m-1}) + d(Tx_{m-1}, Tx_0) \\
                 &\leq d(Tx_0, Tx_1, Tx_{m-1}) + d(Tx_{m-1}, Tx_0) \\
                 &= d(Tx_0, Tx_1, Tx_{m-1}) + d(Tx_1, Tx_0) \\
                 &\leq d(Tx_0, Tx_1, Tx_{m-1}) + \alpha^{m-1}d(Tx_0, Tx_1, Tx_0) \\
                 &\leq d(Tx_0, Tx_1, Tx_{m-1}) + \alpha^{m-1}d(Tx_0, Tx_1, Tx_0) \\
                 &= d(Tx_0, Tx_1, Tx_{m-1}).
\end{align*}
i.e.
\begin{align*}
    d(Tx_0, Tx_1, Tx_m) &\leq d(Tx_0, Tx_1, Tx_{m-1}) \\
                  &\leq d(Tx_0, Tx_1, Tx_{m-2}) \\
                  \cdots \\
\end{align*}

In this way
\begin{align*}
    d(Tx_0, Tx_1, Tx_m) &\leq d(Tx_0, Tx_1, Tx_1) = 0.
\end{align*}

Hence \(d(Tx_0, Tx_1, Tx_m) = 0\).

As \(n, m \to \infty\) in eq (2), we have
\[d(Tx_n, Tx_m, a) \to 0.\]

Thus \(\{Tx_n\}\) is a Cauchy sequence in the complete 2-metric space, the sequence \(\{Tx_n\}\), \(n \in \mathbb{N}\) converges to some \(u \in X\).

\[\lim_{n \to \infty} Tx_n = u = \lim_{n \to \infty} Tx_{n+1} = \lim_{n \to \infty} Ex_n.\]
Now $TEx_n = ETx_n$ and continuity of $T, E$ (using $n \to \infty$) yield $Tu = Eu$. We claim that $Tu = u$.

\[
\begin{align*}
d^2(Tu, u, a) &= \lim_{n \to \infty} d^2(Eu, Ex_n, a) \\
&\leq \lim_{n \to \infty} \left[ \alpha d(Tu, Eu, a)d(Tx_n, Ex_n, a) + \beta d(Tu, Eu, a)d(Tx_n, Eu, a) + \gamma d(Tx_n, Ex_n, a)d(Tx_n, Eu, a) + \delta d(Tx_n, Eu, a)d(Tx_n, Eu, a) \right] \\
&= \alpha d(Tu, Tu, a)d(u, u, a) + \beta d(Tu, Tu, a)d(u, Tu, a) + \gamma d(u, u, a)d(v, u, a) + \delta d(u, v, a)d(v, u, a) \\
&= \delta d^2(Tu, u, a).
\end{align*}
\]

Since $\delta < 1$, we have $d^2(Tu, u, a) = 0 \Rightarrow d(Tu, u, a) = 0$ for all $a \in X$. This implies that $Tu = u$. Therefore $Tu = Eu = u$. Thus $u$ is common fixed point of $E$ and $T$.

**Uniqueness:**

For the uniqueness of the common fixed point. Consider $u, v$ as common fixed points of $E$ and $T$; so $d(u, v, a) = d(Eu, Ev, a)$. Then from (ii)

\[
\begin{align*}
d^2(u, v, a) &= d^2(Eu, Ev, a) \\
&\leq \alpha d(Tu, Eu, a)d(Tv, Ev, a) + \beta d(Tu, Eu, a)d(Tv, Eu, a) + \gamma d(Tv, Ev, a)d(Tv, Eu, a) + \delta d(Tv, Eu, a)d(Tv, Eu, a) \\
&= \alpha d(u, u, a)d(v, v, a) + \beta d(u, u, a)d(v, u, a) + \gamma d(v, v, a)d(v, u, a) + \delta d(u, v, a)d(v, u, a) \\
&= \delta d^2(u, v, a).
\end{align*}
\]

So $d^2(u, v, a) \leq \delta d^2(u, v, a)$.

This gives $d^2(u, v, a) = 0$ i.e. $d(u, v, a) = 0$ for all $a \in X$, since $\delta < 1$. Hence $u = v$ which show that $E$ and $T$ have a common fixed point.

**Theorem 3.3** Let $(X, d)$ be a complete 2-metric space. Let $E, F$ and $T$ be three continuous self mappings of $X$, satisfying the conditions:

(i) $ET = TE, FT = TF, E(X) \subset T(X), F(X) \subset T(X)$

(ii) 

\[
\begin{align*}
d^2(Ex, Fy, a) &\leq \alpha d(Tx, Ex, a)d(Ty, Fy, a) + \beta d(Tx, Ex, a)d(Ty, Ex, a) + \gamma d(Ty, Fy, a)d(Ty, Ex, a) + \delta d(Ty, Fy, a)d(Ty, Ex, a),
\end{align*}
\]

for all $x, y, a \in X$ and $\alpha, \beta, \gamma \geq 0$ with $\max\{\alpha, \delta\} < 1$. Then $E, F$ and $T$ have a unique common fixed point in $X$. 
Proof:
Let \( x_0 \) be a point in \( X \). Since \( E(X) \subset T(X) \), we can choose a point \( x_1 \) in \( X \) such that \( Tx_1 = Ex_0 \), also \( F(X) \subset T(X) \). We can choose a point \( x_2 \) in \( X \) such that \( Tx_2 = Fx_1 \). In general we can choose the point

\[
\begin{align*}
Tx_{2n+1} &= Ex_{2n} \\
Tx_{2n+2} &= Fx_{2n+1}.
\end{align*}
\] (3)

\[
\begin{align*}
Tx_{2n+1} &= Ex_{2n} \\
Tx_{2n+2} &= Fx_{2n+1}.
\end{align*}
\] (4)

For every \( n \in N \), we have

\[
d^2(Tx_{2n+1}, Tx_{2n+2}, a) = d^2(Ex_{2n}, Fx_{2n+1}, a) \\
\leq \alpha d(Tx_{2n}, Ex_{2n}, a)d(Tx_{2n+1}, Fx_{2n+1}, a) \\
+ \beta d(Tx_{2n}, Ex_{2n}, a)d(Tx_{2n+1}, Ex_{2n}, a) \\
+ \gamma d(Tx_{2n+1}, Fx_{2n+1}, a)d(Tx_{2n+1}, Ex_{2n}, a) \\
+ \delta d(Tx_{2n}, Fx_{2n+1}, a)d(Tx_{2n+1}, Ex_{2n}, a) \\
= \alpha d(Tx_{2n}, Tx_{2n+1}, a)d(Tx_{2n+1}, Tx_{2n+2}, a) \\
+ \beta d(Tx_{2n}, Tx_{2n+1}, a)d(Tx_{2n+1}, Tx_{2n+1}, a) \\
+ \gamma d(Tx_{2n+1}, Tx_{2n+2}, a)d(Tx_{2n+1}, Tx_{2n+1}, a) \\
+ \delta d(Tx_{2n}, Tx_{2n+1}, a)d(Tx_{2n+1}, Tx_{2n+1}, a).
\]

Thus, \( d^2(Tx_{2n+1}, Tx_{2n+2}, a) \leq \alpha d(Tx_{2n}, Tx_{2n+1}, a)d(Tx_{2n+1}, Tx_{2n+2}, a) \)
i.e. \( d(Tx_{2n+1}, Tx_{2n+2}, a) \leq \alpha d(Tx_{2n}, Tx_{2n+1}, a), \) for \( n = 1, 2, 3, \cdots \). Similarly we have

\[
d(Tx_{2n}, Tx_{2n+1}, a) \leq \alpha d(Tx_{2n-1}, Tx_{2n}, a) \quad \text{etc.}
\]

Hence

\[
d(Tx_{2n+1}, Tx_{2n+2}, a) \leq \alpha^n d(Tx_1, Tx_0, a), \quad \text{for all } n \in N, \ a \in X.
\]

As same procedure in equation (2) we have the sequence \( \{Tx_n\} \) as a Cauchy sequence in complete 2-metric space. This implies the sequence \( \{Tx_n\}, n \in N \) converges to some \( u \) in \( X \). So by (3) and (4), \( (Ex_{2n}), n \in N \), and \( (Fx_{2n+1}), n \in N \) also converges to the some point \( u \). Thus

\[
\lim_{n \to \infty} Tx_n = u = \lim_{n \to \infty} Ex_{2n} = \lim_{n \to \infty} Fx_{2n+1}.
\]

Using \( ET = TE, FT = TF \) and continuity of \( E, F \) and \( T \), we obtain

\[
Fu = Tu = Eu
\] (5)

Thus

\[
E(Eu) = E(Tu) = T(Eu) = T(Fu) = F(Tu) = F(Eu)
\] (6)
So by (3), (4), (5) and (6) we have,
\[ d^2(Eu, F(Eu), a) \leq \alpha d(Tu, Eu, a)d(T(Eu), F(Eu), a) \\
+ \beta d(Tu, Eu, a)d(T(Eu), Eu, a) \\
+ \gamma d(T(Eu), F(Eu), a)d(T(Eu), Eu, a) \\
+ \delta d(Tu, F(Eu), a)d(T(Eu), Eu, a) \\
= \alpha d(Eu, Eu, a)d(T(Eu), F(Eu), a) \\
+ \beta d(Eu, Eu, a)d(T(Eu), Eu, a) \\
+ \gamma d(T(Eu), F(Eu), a)d(T(Eu), Eu, a) \\
+ \delta d(Eu, F(Eu), a)d(F(Eu), Eu, a) \\
= \delta d(Eu, F(Eu), a)d(Eu, F(Eu), a) \\
= \delta d^2(Eu, F(Eu), a) \]
i.e. \( d^2(Eu, F(Eu), a) \leq \delta d(Eu, F(Eu), a)d(Eu, F(Eu), a) \)
\( \Rightarrow d(Eu, F(Eu), a) \leq \delta d(Eu, F(Eu), a) \), which gives \( d(Eu, F(Eu), a) = 0 \) for all \( a \in X \), since \( \delta < 1 \). Hence
\[ Eu = F(Eu) \] (7)
From (6) & (7) we have \( Eu = F(Eu) = T(Eu) = E(Eu) \). Hence \( Eu \) is a common fixed point of \( E, F \) and \( T \).

**Uniqueness:**
Let \( x \) and \( y \) be two common fixed points of \( E, F \) and \( T \). So \( d(x, y, a) = d(Ex, Fy, a) \). Then by (ii), we have
\[ d^2(x, y, a) = d^2(Ex, Fy, a) \]
\[ \leq \alpha d(Tx, Fx, a)d(Ty, Fy, a) + \beta d(Tx, Ex, a)d(Ty, Ey, a) \\
+ \gamma d(Ty, Fy, a)d(Ty, Ex, a) + \delta d(Tx, Fy, a)d(Ty, Ex, a) \]
So \( d^2(x, y, a) \leq \delta d^2(x, y, a) \). Since \( \delta < 1 \). We have \( d^2(x, y, a) = 0 \) \( \Rightarrow d(x, y, a) = 0 \). Hence \( x = y \). So \( E, F \) and \( T \) have unique common fixed point.

**Theorem 3.4** Let \( (X, d) \) be a complete 2-metric space. Let \( E, F \) and \( T \) be three continuous self mappings of \( X \), satisfying the conditions:

(i) \( ET = TE, FT = TF, E(X) \subset T(X), F(X) \subset T(X) \)

(ii) \[ d^2(E^p x, F^q y, a) \leq \alpha d(Tx, E^p x, a)d(TEy, F^q y, a) + \beta d(Tx, E^p x, a)d(TEy, E^p x, a) \\
+ \gamma d(Ty, F^q y, a)d(Ty, E^p x, a) + \delta d(Tx, F^q y, a)d(Ty, E^p x, a) \]

For all \( x, y, a \in X \), \( Tx \neq Ty \) and \( \alpha, \beta, \gamma \geq 0 \) with \( \max\{\alpha, \delta\} < 1 \). If some positive integer \( p, q \) exist such that \( E^p, F^q \) and \( T \) are continuous, Then \( E, F \) and \( T \) have a unique common fixed point in \( X \).
Proof:

It follows from (i)

\[ E^p T = T E^p, F^q T = T F^q, E^p (X) \subset T (X) \quad \text{and} \quad F^q (X) \subset T (X) \] (8)

By theorem (3.2), we have \( T, E^p \) and \( F^q \) have a unique common fixed \( u \in X \). Then

\[ T (E u) = E (T u) = E (u) = E (E^p u) = E^p (E u) \] (9)

and

\[ T (F u) = F (T u) = F (u) = F (F^q u) = F^q (F u). \] (10)

By (ii)

\[
\begin{align*}
&d^2(E u, F u, a) = d^2(E^p(E u), F^q(F u), a) \\
&\quad \leq \alpha d(T(E u), E^p(E u), a)d(T E(F u), F^q(F u), a) \\
&\quad + \beta d(T(E u), E^p(E u), a)d(T E(F u), F^q(F u), a) \\
&\quad + \gamma d(T(F u), F^q(F u), a)d(T(F u), E^p(E u), a) \\
&\quad + \delta d(T(E u), F^q(F u), a)d(T(F u), E^p(E u), a) \\
&\quad = \delta d^2(E u, F u, a) \quad \text{for all} \ a \in X.
\end{align*}
\]

By (9) and (10)

\[ \Rightarrow d(E u, F u, a) = 0 \quad \text{for all} \ a \in X, \ \text{since} \ \delta < 1. \]

This shows that \( E u = F u \). Hence \( E u \) is common fixed point of \( T, E^p \) and \( F^q \) is a common fixed point of \( T \) and \( F^q \). The uniqueness of \( u \), from (9) and (10) implies \( u = E u = F u = T u \). This completes the proof of the theorem.

**Theorem 3.5** Let \((X, d)\) be a complete 2-metric space. Let \( E, F \) and \( T \) be three continuous self mappings of \( X \), satisfying the conditions:

(i) \( E F T = T E F, F E T = T E F, E F (X) \subset T (X), F E (X) \subset T (X) \)

(ii)

\[
\begin{align*}
&d^2(E F x, F E y, a) \leq \alpha d(T x, E F y, a)d(T y, F E y, a) + \beta d(T x, E F x, a)d(T y, E F x, a) \\
&\quad + \gamma d(T y, F E y, a)d(T y, E F x, a) + \delta d(T x, F E y, a)d(T y, E F x, a).
\end{align*}
\]

For all \( x, y, a \in X \), with \( T x \neq T y \) and \( \alpha, \beta, \gamma \geq 0 \) with \( \alpha < 1, 0 \leq \delta < 1 \). Further if \( T, E F \) and \( F E \) are continuous, then \( E, F \) and \( T \) have a unique common fixed point in \( X \).
Proof: - Let $EF = S_1$ and $FE = S_2$ then by (ii)

$$d^2(S_1x, S_2y, a) \leq \alpha d(Tx, S_1y, a)d(Ty, S_2y, a)$$
$$+ \beta d(Tx, S_1x, a)d(Ty, S_1x, a)$$
$$+ \gamma d(Ty, S_2y, a)d(Ty, S_1x, a)$$
$$+ \delta d(Tx, S_2y, a)d(TyS_1x, a),$$

For all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha < 1, 0 \leq \delta < 1$. And conditions $S_2T = TS_2, S_1T = TS_1, S_1(X) \subset T(X)$ and $S_2(x) \subset T(X)$ are satisfied. Further $T, S_1$ and $S_2$ are continuous self mapping of $X$. Therefore by theorem (3.3), there exists a unique fixed point $u$ such that $u = S_1u = S_2u = Tu$ also, $Eu = E(S_2u) = EF(Eu) = S_1(Eu)$ and $Fu = F(S_1u) = FE(Fu) = S_2(Fu)$. This means that $Eu$ is a fixed point of $S_1$ and $Fu$ is a fixed point of $S_2$. The uniqueness of $u$ implies $u = Tu = Eu = Fu$. This completes the proof of the theorem.

References


Received: July 2, 2008