A Note on Radius of Starlikeness and Convexity of $p$-Valent Analytic Functions

S. Latha

Department of Mathematics
Yuvaraja’s College, University of Mysore
Mysore - 570 005, India
drlatha@gmail.com

D. S. Raju

Department of Mathematics
Vidyavardhaka College of Engineering
Mysore - 570 002, India
rajudsvm@gmail.com

N. Poornima

Department of Mathematics
Yuvaraja’s College, University of Mysore
Mysore - 570 005, India
poornimn@gmail.com

Abstract. Let $\mathcal{A}_p$ denote the class of analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, ...\})$$

defined in the unit disc $U = \{z : |z| < 1\}$ and $\Omega$ denote the class of functions such that $\omega(0) = 0$ and $|\omega(z)| < 1$. Let $\mathcal{P}(A, B, p, \alpha)$ be the class of functions of the form

$$p(z) = p + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad p(z) = \frac{p + \gamma \omega(z)}{1 + B \omega(z)}, \quad -1 \leq B < A \leq 1$$

where $\gamma = (p - \alpha) A + \alpha B$. In this paper, we define the class $\mathcal{S}_q(A, B, p, \alpha)$ of functions $f(z) \in \mathcal{A}_p$ such that

$$q + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = p(z) \quad \text{for} \quad p(z) \in \mathcal{P}(A, B, p, \alpha)$$
and radius of starlikeness and convexity of functions in this class are studied.

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1. Introduction

Let \( A_p \) denote the class of analytic functions of the form

\[ f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \]

defined in the unit disc \( U = \{ z : |z| < 1 \} \). Let \( \Omega \) denote the class of bounded analytic functions \( \omega(z) \) in \( U \) satisfying the conditions \( \omega(0) = 0 \) and \( |\omega(z)| \leq 1 (z \in U) \).

For functions \( g(z) \) and \( G(z) \) analytic in \( U \), we say that \( g(z) \) is subordinate to \( G(z) \) if there exists a Schwarz function \( \omega(z) \in \Omega \) such that

\[ g(z) = G(\omega(z)) \]

If \( G(z) \) is univalent in \( U \), then \( g(z) \) is subordinate to \( G(z) \) if and only if \( g(0) = G(0) \) and \( g(U) \subset G(U) \).

For \( -1 \leq B < A \leq 1 \) and \( 0 \leq \alpha < p \), \( \mathcal{P}(A, B, p, \alpha) \) [1] denote the class of analytic functions defined in \( U \) such that

\[ p(z) = p + \sum_{n=1}^{\infty} a_n z^n \text{ and } p(z) = \frac{p + \gamma \omega(z)}{1 + B \omega(z)}, -1 \leq B < A \leq 1. \]

where \( \gamma = (p - \alpha)A + \alpha B \). Further, \( p(z) \in \mathcal{P}(A, B, p, \alpha) \) if and only if

\[ p(z) = (p - \alpha)p_1(z) + \alpha, \quad p_1(z) \in \mathcal{P}(A, B) \]

where \( \mathcal{P}(A, B) \) [3] is the Janowski class of functions \( p_1(z) \) which are of the form

\[ p_1(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \]

and are analytic in \( U \), such that \( p_1(z) \in \mathcal{P}(A, B) \) if and only if

\[ p_1(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)}, -1 \leq B < A \leq 1, \omega(z) \in \Omega, z \in U. \]

We define the class \( \mathcal{S}_q(A, B, p, \alpha) \) of functions \( f(z) \in A_p \) such that

\[ q + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = p(z) \text{ for } p(z) \in \mathcal{P}(A, B, p, \alpha) \]
2. MAIN RESULTS

**Lemma 2.1. (Jack’s Lemma)** [2]: Let \( \omega(z) \) be a regular function in the unit disc \( \mathcal{U} \) with \( \omega(0) = 0 \), then if \( |\omega(z)| \) attains its maximum value on the circle \( |z| = r \) at a point \( z_1 \), we can write \( z_1 \omega'(z_1) = k \omega(z_1) \), where \( k \) is real and \( k \geq 1 \).

**Lemma 2.2.** The function

\[
\omega = \begin{cases} 
\frac{p + \gamma z}{1 + Bz} & \text{for } B \neq 0, \\
p + \gamma z & \text{for } B = 0
\end{cases}
\]

maps \( |z| = r \) onto a disc centered at \( C(r) \) and having the radius \( \rho(r) \) given by

\[
C(r) = \begin{cases} 
\left( \frac{p - \gamma Br^2}{1 - B^2r^2}, 0 \right) & \text{for } B \neq 0, \\
(p, 0) & \text{for } B = 0
\end{cases}
\]

and

\[
\rho(r) = \begin{cases} 
\frac{(\gamma - pB)r}{1 - B^2r^2} & \text{for } B \neq 0, \\
|\gamma|r & \text{for } B = 0.
\end{cases}
\]

**Proof.** Consider

\[
\omega = \frac{p + \gamma z}{1 + Bz} \iff z = \frac{\omega - p}{\gamma - B\omega} \iff |z|^2 = r^2 = \frac{|\omega - p|^2}{|\gamma - B\omega|^2} \quad \text{for } B \neq 0
\]

\[
\Rightarrow u^2 + v^2 + \left( \frac{2\gamma Br^2 - 2p}{1 - B^2r^2} \right) u + \frac{p^2 - \gamma^2 r^2}{1 - B^2r^2} = 0, \quad \text{for } B \neq 0.
\]

\[
\omega = 1 + \gamma z \iff z = \frac{\omega - p}{\gamma} \iff |z|^2 = r^2 = \frac{|\omega - p|^2}{|\gamma|^2} \quad \text{for } B = 0
\]

\[
\Rightarrow u^2 + v^2 - 2u + p^2 - \gamma^2 r^2 = 0, \quad \text{for } B = 0.
\]

Lemma follows from (2.4).

**Lemma 2.3.** The function

\[
\omega = \begin{cases} 
\frac{(\gamma - pB)z}{1 + Bz} & \text{for } B \neq 0, \\
\gamma z & \text{for } B = 0
\end{cases}
\]
maps \(|z|=r\) onto a disc centered at \(C(r)\) and having the radius \(\rho(r)\) given by

\[
C(r) = \begin{cases}
  \left( \frac{-B(\gamma - pB)r^2}{1 - B^2r^2}, 0 \right) & \text{for } B \neq 0, \\
  (0, 0) & \text{for } B = 0
\end{cases}
\]

and

\[
\rho(r) = \begin{cases}
  \frac{(\gamma - pB)r^2}{1 - B^2r^2} & \text{for } B \neq 0, \\
  |\gamma|r & \text{for } B = 0.
\end{cases}
\]

**Proof.** Consider

\[
\begin{cases}
  \omega = \frac{\gamma - pB}{1 + Bz} & \Leftrightarrow \ z = \frac{\omega}{\gamma - pB - B\omega} & \Rightarrow |z|^2 = r^2 = \frac{|\omega|^2}{|\gamma - pB - B\omega|^2} \quad \text{for } B \neq 0 \\
  u^2 + v^2 + \left( \frac{2B(\gamma - pB)r^2}{1 - B^2r^2} \right) u + \left( \frac{(\gamma - pB)^2r^2}{1 - B^2r^2} \right) = 0, \quad \text{for } B \neq 0.
\end{cases}
\]

\[
\omega = \gamma z \quad \Leftrightarrow \ z = \frac{\omega}{\gamma} \quad \Rightarrow |z|^2 = r^2 = \frac{|\omega|^2}{|\gamma|^2} \quad \Rightarrow u^2 + v^2 - \gamma^2 r^2 = 0 \quad \text{for } B = 0.
\]

Lemma follows from (2.8). \(\square\)

**Theorem 2.4.** Let \(f(z) \in A_p\) be such that

\[
\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q < \begin{cases}
  \frac{(\gamma - pB)z}{1 + Bz} = F_1(z) & \text{for } B \neq 0, \\
  \gamma z = F_2(z) & \text{for } B = 0.
\end{cases}
\]

Then, \(f(z) \in S_q(A, B, p, \alpha)\) and this result is sharp being obtained by the function \(\frac{p + \gamma z}{1 + Bz}\).

**Proof.** Define

\[
\frac{f^{(q)}(z)}{z^{p-q}} = \begin{cases}
  (1 + B\omega(z))\left(\frac{\gamma - pB}{B}\right) & \text{for } B \neq 0, \\
  e^{\gamma \omega(z)} & \text{for } B = 0
\end{cases}
\]

where \((1 + B\omega(z))\left(\frac{\gamma - pB}{B}\right)\) and \(e^{\gamma \omega(z)}\) have the value at 1 at the origin. Then \(\omega(z)\) is analytic in \(U\) and \(\omega(0) = 0\). On logarithmic differentiation we
get,

\[
(2.11) \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q < \begin{cases} \\
\frac{(\gamma - pB)z\omega'(z)}{1 + B\omega(z)} & \text{for } B \neq 0, \\
\gamma z\omega'(z) & \text{for } B = 0.
\end{cases}
\]

Now it is easy to realize that subordination (2.9) is equivalent to \(|\omega(z)| < 1\) for all \(z \in U\). By Jack’s Lemma it follows that, there exists a point \(z_1 \in U\) such that

\[
(2.12) \quad \frac{z_1f^{(q+1)}(z_1)}{f^{(q)}(z_1)} - p + q < \begin{cases} \\
\frac{(\gamma - pB)k\omega(z_1)}{1 + B\omega(z_1)} = F_1(\omega(z_1)) \notin F_1(U) & \text{for } B \neq 0, \\
\gamma k\omega(z_1) = F_2(\omega(z_1)) \notin F_2(U) & \text{for } B = 0.
\end{cases}
\]

This contradicts our assumption given by (2.9) and the fact that \(|\omega(z)| < 1\) for all \(z \in U\).

By using the condition (2.11), we get

\[
(2.13) \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + q = \begin{cases} \\
p + \gamma \omega(z) \quad & \text{for } B \neq 0, \\
p + \gamma \omega(z) & \text{for } B = 0.
\end{cases}
\]

Now by inequality (2.13) we obtain

\[
(2.14) \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + q < \begin{cases} \\
p + \gamma z \quad & \text{for } B \neq 0, \\
p + \gamma z & \text{for } B = 0.
\end{cases}
\]

By inequality (2.14) it follows that \(f(z) \in S_q(A, B, p, \alpha)\).

**Corollary 2.5.** Let \(f(z) \in S_q(A, B, p, \alpha)\). Then, \(f(z)\) can be written in the form

\[
(2.15) \quad f_s^{(q)}(z) = \begin{cases} \\
z^{p-q}(1 + B\omega(z))^{\frac{\gamma-pB}{B}} & \text{for } B \neq 0, \\
z^{p-q}e^{\gamma\omega(z)} & \text{for } B = 0.
\end{cases}
\]

**Theorem 2.6.** The radius of starlikeness and the radius of convexity of the \(S_q(A, B, p, \alpha)\) is given by

\[
(2.16) \quad R_{sc} = \frac{2(p-q)}{(\gamma - pB) + \sqrt{(\gamma - pB)^2 + 4(p-q)\left[(\gamma - pB) + (p-q)B^2\right]}}
\]
The radius is sharp, being attained by the function

\[ f_s^{(q)}(z) = \begin{cases} 
  z^{p-q}(1 + B\omega(z))^{\frac{\gamma-pB}{B}} & \text{for } B \neq 0, \\
  z^{p-q}e^{\gamma\omega(z)} & \text{for } B = 0.
\]  

(2.17)

**Proof.** By Lemma 2.2, the set of values \( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \) is obtained which comprises the closed disc with center at \( C(r) \) and the radius \( \rho(r) \), where

\[ C(r) = \frac{(p-q) - [B(\gamma - pB) + (p-q)B^2] r^2}{1 - B^2 r^2} \quad \text{and} \quad \rho(r) = \frac{(\gamma-pB)r}{1 - B^2 r^2}. \]

(2.18)

Now by the definition of the class \( S_q(A, B, p) \) we have,

\[ \left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - C(r) \right| \leq \rho(r). \]

(2.19)

This gives,

\[ \Re \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right) \geq \frac{(p-q) - (\gamma - pB)r - [B(\gamma - pB) + (p-q)B^2] r^2}{1 - B^2 r^2}. \]

Hence for \( r < R_{sc} \) the right hand side of the preceding inequality is positive, implying that

\[ R_{sc} \leq \frac{2(p-q)}{(\gamma-pB) + \sqrt{(\gamma-pB)^2 + 4(p-q) [(\gamma-pB) + (p-q)B^2]}}. \]

(2.21)

The radius is sharp, being attained by the function \( f_s^{(q)}(z) \) given by (2.17).

**Remark 2.7.** For parametric values of \( A, B, p \) and \( \alpha \) we get the well known results proved by Aouf, Nasr and also the results of Yasar Polatoglu.

**References**


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