Fixed Point Theorems for Condensing Maps

in $S$-KKM Class

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Abstract

This paper presents some new fixed point results for condensing multimaps in $s$-KKM class in the setting of locally $G$-convex uniform spaces. We mainly show that every l.s.c., closed and generalized condensing self-multimap with $s$-KKM property on a complete locally $G$-convex uniform space has a fixed point. Some applications to quasi-equilibrium problem of these fixed point results are also given.

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1. Introduction and Preliminaries

Chang et al. [3] introduced the $S$-KKM class, where a lot of interesting fixed point theorems on locally convex topological vector spaces were established. Later, these results were extended to locally $G$-convex uniform spaces in [8]. On the other hand, we introduced the concepts of measure of precompactness and condensing multimaps on locally $G$-convex uniform spaces in [5]. The intent of this paper is to present some fixed point theorems for condensing multimaps in $S$-KKM class in the setting of locally $G$-convex uniform spaces. These results partially cover our previous results in [3], [4] and [8].

We now recall some basic definitions and facts. For a nonempty set $Y$, $2^Y$ denotes the class of all subsets of $Y$ and $\langle Y \rangle$ denotes the class of all nonempty finite subsets of $Y$. A multimap $T : X \to 2^Y$ is a function from a set $X$ into the power set $2^Y$ of $Y$. The notation $T : X \to Y$ stands for a multimap $T : X \to 2^Y$ having nonempty values.

For a multimap $T : X \to 2^Y$, $A \subseteq X$ and $B \subseteq Y$, the image of $A$ under $T$ is the set $T(A) = \bigcup_{x \in A} T(x)$; and the inverse image of $B$ under $T$ is $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$.

All topological spaces are supposed to be Hausdorff. The closure of a subset $X$ of a topological space is denoted by $\overline{X}$. Let $X$ and $Y$ be two topological spaces. A multimap $T : X \to 2^Y$ is said to be
(a) lower semicontinuous (l.s.c.) if $T^{-1}(B)$ is open in $X$ for each open subset $B$ of $Y$;
(b) compact if $T(X)$ is contained in a compact subset of $Y$;
(c) closed if its graph $Gr(T) = \{(x, y) : y \in T(x), x \in X\}$ is a closed subset of $X \times Y$.

For $n \geq 0$, $\Delta_n$ denotes the standard $n$-simplex of $\mathbb{R}^{n+1}$, that is,
\[\Delta_n = \left\{ \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1} : \alpha_i \geq 0 \text{ for all } i \text{ and } \sum_{i=0}^{n} \alpha_i = 1 \right\};\]
and $\{e_0, \ldots, e_n\}$, the standard basis of $\mathbb{R}^{n+1}$, is the set of the vertices of $\Delta_n$.

Definition 1.1. (Park and Kim [10]) A generalized convex space or a $G$-convex space $(E; \Gamma)$ consists of a topological space $E$ and a map $\Gamma : \langle E \rangle \to E$ such
that

(a) for any \(A, B \in \langle E \rangle\), \(A \subseteq B\) implies \(\Gamma(A) \subseteq \Gamma(B)\); and

(b) for each \(A = \{a_0, \ldots , a_n\} \in \langle E \rangle\) with \(|A| = n + 1\), there exists a continuous function \(\varphi_A : \Delta_n \rightarrow \Gamma(A)\) such that if \(0 \leq i_0 < i_1 < \cdots < i_k \leq n\), then \(\varphi_A(\text{co}\{e_{i_0}, \ldots , e_{i_k}\}) \subseteq \Gamma(\{a_{i_0}, \ldots , a_{i_k}\})\).

In this paper, we assume that a \(G\)-convex space \((E; \Gamma)\) always satisfies the extra condition: \(x \in \Gamma(\{x\})\) for any \(x \in E\).

A subset \(K\) of a \(G\)-convex space \((E; \Gamma)\) is said to be \(\Gamma\)-convex if for any \(A \in \langle K \rangle\), \(\Gamma(A) \subseteq K\). For a nonempty subset \(Q\) of \(E\), the \(\Gamma\)-convex hull of \(Q\), denoted by \(\Gamma\)-co\((Q)\), is defined by

\[\Gamma\text{-co}(Q) = \cap \{C : Q \subseteq C \subseteq E, C \text{ is } \Gamma\text{-convex}\}\].

It is easy to see that \(\Gamma\text{-co}(Q)\) is the smallest \(\Gamma\)-convex subset of \(E\) containing \(Q\). For convenience, we also express \(\Gamma(A)\) by \(\Gamma_A\). By the definition of a \(G\)-convex space \((E; \Gamma)\), it is easy to check that \(A \subseteq \Gamma_A\) for any \(A \in \langle E \rangle\). If \(X\) is a nonempty \(\Gamma\)-convex subset of \(E\), then \((X; \Gamma|_{\langle X \rangle})\) is also a \(G\)-convex space.

The concept of \(S\)-KKM property in \(G\)-convex spaces was introduced in [8]:

**Definition 1.2.** Let \(X\) be a nonempty set, \((E; \Gamma)\) a \(G\)-convex space and \(Y\) a topological space. If \(S : X \rightarrow E\), \(T : E \rightarrow Y\) and \(F : X \rightarrow Y\) are three multimaps satisfying

\[T(\Gamma_{S(A)}) \subseteq F(A)\]

for any \(A \in \langle X \rangle\), then \(F\) is called a \(S\)-KKM mapping with respect to \(T\).

A multimap \(T : E \rightarrow Y\) is said to have the \(S\)-KKM property if for any \(S\)-KKM mapping \(F\) with respect to \(T\), the family \(\{F(x) : x \in X\}\) has the finite intersection property. The class \(\mathcal{S}\text{-KKM}(X, E, Y)\) is defined to be the set \(\{T : E \rightarrow Y : T\text{ has the }\mathcal{S}\text{-KKM property}\}\).

In the case that \(X = E\) and \(S\) is the identity mapping \(1_X\), \(\mathcal{S}\text{-KKM}(X, E, Y)\) is abbreviated as \(\text{KKM}(E, Y)\), and a \(1_X\)-KKM mapping with respect to \(T\) is called a KKM mapping with respect to \(T\), and \(1_X\)-KKM property is called KKM property.

For any nonempty set \(X\), \(G\)-convex space \((E; \Gamma)\), topological space \(Y\) and any \(S : X \rightarrow E\), one has \(\text{KKM}(E, Y) \subseteq \mathcal{S}\text{-KKM}(X, E, Y)\).
A uniformity for a set $X$ is a nonempty family $\mathcal{U}$ of subsets of $X \times X$ such that
(a) each member of $\mathcal{U}$ contains the diagonal $\Delta$;
(b) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
(c) if $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some $V$ in $\mathcal{U}$;
(d) if $U$ and $V$ are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$; and
(e) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

Every member $V$ in $\mathcal{U}$ is called an entourage. An entourage $V$ is said to symmetric if $(x, y) \in V$ whenever $(y, x) \in V$.

If $(X, U)$ is a uniform space, then the topology $\mathcal{T}$ induced by $\mathcal{U}$ is the family of all subsets $W$ of $X$ such that for each $x$ in $W$ there is $U$ in $\mathcal{U}$ such that $U[x] \subseteq W$, where $U[x]$ is defined as $\{y \in X : (x, y) \in U\}$. If $H$ is a subset of $X$ and $U$ is in $\mathcal{U}$, then $U[H] := \cup_{x \in H} U[x]$. For details of uniform spaces we refer to [6].

**Definition 1.3.** (Watson [12]) A $G$-convex uniform space $(E; \mathcal{U}, \Gamma)$ is a $G$-convex space so that its topology is induced by a uniformity $\mathcal{U}$. A G-convex uniform space $(E; \mathcal{U}, \Gamma)$ is said to be a locally $G$-convex uniform space if the uniformity $\mathcal{U}$ has a base $\mathcal{B}$ consisting of open symmetric entourages such that for each $V \in \mathcal{B}$
(1.3.1) $V[x]$ is $\Gamma$-convex for any $x \in E$; and
(1.3.2) $V[K]$ is $\Gamma$-convex whenever $K$ is a $\Gamma$-convex subset of $X$.

A subset $S$ of a uniform space $E$ is said to be precompact if, for any entourage $V$, there is a finite subset $N$ of $E$ such that $S \subseteq V[N]$. In this paper, for a locally $G$-convex uniform space $(E; \mathcal{U}, \Gamma)$, the convex structure $\Gamma$ is assumed to have the property that $\Gamma$-co($A$) is precompact whenever $A$ is precompact, and $\mathcal{B}$ will always denotes a base of $\mathcal{U}$ so that it has the properties described in Definition 1.3.

Finally, we state a fixed point theorem of Kuo et al. [8, Theorem 3.9] which will be quoted in the sequel.

**Theorem 1.4.** Let $X$ be any nonempty set, $(E; \mathcal{U}, \Gamma)$ a locally $G$-convex
uniform space and \( s : X \to E \). If \( T \in s\text{-}KKM(X, E, E) \) satisfies that
\begin{enumerate}[(1.4.1)]
  \item \( T \) is compact and closed;
  \item \( \overline{T(E)} \subseteq s(X) \), then \( T \) has a fixed point.
\end{enumerate}

If \( s \) is surjective, then condition (1.4.2) is surely satisfied, so Theorem 1.4 holds for a surjection \( s \).

2. Fixed Point Theorems

We begin this section by presenting a fixed point theorem of Mönch type, cf. Agarwal and O’Regan [1], [2] and the references therein.

**Theorem 2.1.** Let \( X \) be a nonempty closed \( \Gamma \)-convex subset of a complete locally \( G \)-convex uniform space \((E; \mathcal{U}, \Gamma)\) and \( s : X \to E \) a surjection. Assume \( T \in s\text{-}KKM(X, E, E) \) is closed and satisfies the following conditions:
\begin{enumerate}[(2.1.1)]
  \item \( T \) maps compact sets into precompact sets;
  \item there is \( x_0 \in X \) such that if \( C \subseteq X \) is countable and \( C \subseteq \overline{\Gamma\text{-co}(T(C))\cup \{x_0\}} \), then \( C \) is precompact;
  \item for any precompact subset \( A \) of \( X \) there is a countable subset \( B \) of \( A \) with \( \overline{B} = \overline{A} \);
  \item \( T(\overline{A}) \subseteq \overline{T(A)} \) for any precompact subset \( A \) of \( X \).
\end{enumerate}
Then \( T \) has a fixed point.

**Proof.** Let \( D_0 = \{x_0\} \) and \( D_n = \Gamma\text{-co}(T(D_{n-1}) \cup \{x_0\}) \) for \( n \in \mathbb{N} \), and \( D = \bigcup_{n=0}^{+\infty} D_n \). Noting that each \( D_n \) is \( \Gamma \)-convex and
\[
D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n \cdots \subseteq D \subseteq X,
\]
we deduce that \( D \) is \( \Gamma \)-convex. Also, it is obvious that
\[
D = \bigcup_{n=1}^{+\infty} \Gamma\text{-co}(T(D_{n-1}) \cup \{x_0\}) \subseteq \Gamma\text{-co}(T(D) \cup \{x_0\}). \tag{2.1}
\]
On the other hand, since \( \{D_n\}_{0}^{+\infty} \) is increasing and since
\[
\Gamma\text{-co}(T(D) \cup \{x_0\}) = \bigcup \{\Gamma\text{-co}(A) : A \in T(D) \cup \{x_0\}\}, \tag{2.2}
\]
cf. [11, Lemma 2.1], we have that for any \( A \in \langle T(D) \cup \{x_0\} \rangle \) there is a \( D_k \) with \( A \in \langle T(D_{k-1}) \cup \{x_0\} \rangle \), so

\[
\Gamma\text{-co}(A) \subseteq \Gamma\text{-co}(T(D_{k-1}) \cup \{x_0\}) \subseteq \bigcup_{n=1}^{+\infty} \Gamma\text{-co}(T(D_{n-1}) \cup \{x_0\}) = D. \tag{2.3}
\]

It follows from (2.1), (2.2) and (2.3) that

\[
D = \Gamma\text{-co}(T(D) \cup \{x_0\}). \tag{2.4}
\]

We now show by induction that each \( D_n \) is precompact. Obviously, \( D_0 \) is precompact. Assume \( D_k \) is precompact for \( k \geq 1 \). Condition (2.1.1) guarantees that \( \overline{T(D_k)} \) is precompact, so \( \overline{T(D_k)} \cup \{x_0\} \) is compact. By our hypotheses on locally \( G \)-convex uniform spaces, \( \Gamma\text{-co}(\overline{T(D_k)} \cup \{x_0\}) \) is precompact. Thus \( D_{k+1} \) is precompact once we note that

\[
D_{k+1} = \Gamma\text{-co}(T(D_k) \cup \{x_0\}) \subseteq \Gamma\text{-co}(\overline{T(D_k)} \cup \{x_0\}).
\]

This completes the induction for showing each \( D_n \) is precompact.

Next, condition (2.1.3) implies that for each \( n \in \mathbb{N} \cup \{0\} \), there is a countable subset \( C_n \) of \( D_n \) such that \( \overline{C_n} = \overline{D_n} \). Let \( C = \bigcup_{n=0}^{+\infty} C_n \). Since

\[
\bigcup_{n=0}^{+\infty} D_n \subseteq \bigcup_{n=0}^{+\infty} \overline{D_n} \subseteq \bigcup_{n=0}^{+\infty} D_n,
\]

we have

\[
\bigcup_{n=0}^{+\infty} \overline{D_n} = \bigcup_{n=0}^{+\infty} D_n = \overline{D}
\]

and

\[
\bigcup_{n=0}^{+\infty} \overline{C_n} = \bigcup_{n=0}^{+\infty} C_n = \overline{C}.
\]

Consequently,

\[
\overline{C} = \overline{D}. \tag{2.5}
\]
Combining (2.4) and (2.5) yields that
\[
C \subseteq \overline{C} = \overline{\Gamma\text{-co}(T(D) \cup \{x_0\})} \\
\subseteq \overline{\Gamma\text{-co}(T(D) \cup \{x_0\})} \\
= \overline{\Gamma\text{-co}(T(C) \cup \{x_0\})}. \tag{2.6}
\]
Since \(C\) is countable, it follows from (2.1.2) and (2.6) that \(C\) is compact, and hence \(D\) is compact by (2.5).

Finally, notice that (2.4) implies \(T(D) \subseteq D\), which together with (2.1.4) shows \(T(D) \subseteq D\). Putting \(K = s^{-}(\overline{D})\), we see that \(s : K \to \overline{D}\) is surjective and \(T \in s\text{-KKM}(K, \overline{D}, \overline{D})\) is compact and closed. The existence of a fixed point for \(T\) now follows from Theorem 1.4.

That a multimap \(T : X \to Y\) is l.s.c. could be phrased as: For any \(x \in X\) and any net \(\{x^\alpha\}\) in \(X\) which converges to \(x\) and any \(y \in T(x)\), there exists a net \(\{y^\alpha\}\) in \(Y\) such that for each \(\alpha\), \(y^\alpha \in T(x^\alpha)\) and \(y^\alpha\) converges to \(y\). By means of this characterization, it is easy to see that \(T(A) \subseteq \overline{T(A)}\) for any subset \(A\) of \(X\). Hence, the above theorem holds true when (2.1.4) is replaced with

\[(2.1.4)'\ T \text{ is l.s.c.}\]

**Theorem 2.2.** \(X\) be a nonempty closed \(\Gamma\text{-convex}\) subset of a complete locally \(G\text{-convex}\) uniform space \((E;U,\Gamma)\) and \(s : X \to E\) a surjection. Assume \(T : X \to X\) is closed and satisfies

\[(2.2.1)\ there \ is \ \ x_0 \in X \ such \ that \ if \ A \subseteq X \ with \ A \subseteq \Gamma\text{-co}(T(A) \cup \{x_0\}), \ then \ A \ is \ precompact;\]

\[(2.2.2)\ for \ any \ precompact \ \Gamma\text{-convex} \ subset \ A \ of \ X \ with \ T(x) \cap \overline{A} \neq \emptyset \ for \ each \ x \in \overline{A}, \ the \ multimap \ H : \overline{A} \to \overline{A} \ defined \ by \ H(x) = T(x) \cap \overline{A} \ is \ in \ s\text{-KKM}(s^{-}(\overline{A}), \overline{A}, \overline{A}).\]

Then \(T\) has a fixed point.

**Proof.** Let \(D\) and \(D_n, n \in \mathbb{N} \cup \{0\}\), be as in the proof of Theorem 2.1. We have seen that \(D\) is \(\Gamma\text{-convex}\) and (2.4) holds. Thus, by (2.2.1) and (2.4), \(D\) is precompact. Moreover, it follows from (2.4) that \(T(D) \subseteq D\). Hence, if \(x \in D\), we surely have

\[T(x) \cap \overline{D} \supseteq T(x) \cap D = T(x) \neq \emptyset.\]
If \( x \in \overline{D} \setminus D \), choose a net \( \{x_\alpha\} \) in \( D \) such that \( x_\alpha \to x \). For any \( \alpha \), choose \( y_\alpha \in T(x_\alpha) \). Since \( \{y_\alpha\} \subseteq T(D) \subseteq \overline{D} \) and \( \overline{D} \) is compact, \( y_\alpha \) has a subnet \( y_{\alpha_j} \) so that \( y_{\alpha_j} \to y \) for some \( y \in \overline{D} \). Therefore the multimap \( H : \overline{D} \to \overline{D} \) defined by \( H(x) = T(x) \cap \overline{D} \) is well-defined. Let \( K = s^{-}(\overline{D}) \). Then \( s : K \to \overline{D} \) is surjective. Furthermore, \( H \) is in \( s\text{-KKM}(K, \overline{D}, \overline{D}) \) by (2.2.2). Clearly, \( H \) is compact and closed, so it has a fixed point by Theorem 1.4.

In the remainder of this section we investigate the fixed point problem for condensing multimaps in \( s\text{-KKM} \) class. At first, we quote some definitions and results in [5].

**Definition 2.3.** (Huang et al. [5]) For a subset \( A \) of locally \( G \)-convex uniform space \((E; U, \Gamma)\), the measure of precompactness of \( A \) is defined to be

\[
\Psi(A) = \{ V \in \mathcal{B} : A \subseteq V[S] \text{ for some precompact subset } S \text{ of } E \}.
\]

Clearly, the larger \( \Psi(A) \) the more nearly is \( A \) precompact. In fact, we have

**Proposition 2.4.** (Huang et al. [5]) Let \( A \) and \( B \) be subsets of \((E; U, \Gamma)\). Then
(a) \( A \) is precompact iff \( \Psi(A) = \mathcal{B} \);
(b) \( \Psi(A) \supseteq \Psi(B) \) if \( A \subseteq B \);
(c) \( \Psi(\Gamma\text{-co}(A)) = \Psi(A) \);
(d) \( \Psi(A \cup B) = \Psi(A) \cap \Psi(B) \).

**Definition 2.5.** (Huang et al. [5]) Suppose \( X \) is a nonempty subset of a locally \( G \)-convex uniform space \((E; U, \Gamma)\) and \( \Psi \) is the measure of precompactness in Definition 2.3. A multimap \( T : X \to 2^E \) is called condensing provided \( \Psi(A) \not\supseteq \Psi(T(A)) \) for any subset \( A \) that is not precompact. \( T \) is called generalized condensing if whenever \( A \subseteq X \), \( T(A) \subseteq A \) and \( A \setminus \Gamma\text{-co}(T(A)) \) is precompact, then \( A \) is precompact.

It is obvious that every compact map or every map defined on a compact set is condensing. Also, every condensing map is generalized condensing.

**Lemma 2.6.** (Huang et al. [5]) Let \( X \) be a nonempty \( \Gamma \)-convex subset of a locally \( G \)-convex uniform space \((E; U, \Gamma)\) and \( T : X \to X \). Then,
(a) for any \( x_0 \in X \), there is a precompact \( \Gamma \)-convex subset \( K \) of \( X \) such that
\[
T(K) \subseteq K \text{ and } K = \Gamma\text{-co}(T(K) \cup \{x_0\}).
\]
provided that $T$ is condensing;

(b) there is a nonempty precompact $\Gamma$-convex subset $K$ of $X$ such that $T(K) \subseteq K$ and

$$T(x) \cap \overline{K} \neq \emptyset \text{ for any } x \in \overline{K}.$$  

provided that $X$ is complete and $T$ is generalized condensing and closed.

We now in a position to present fixed point results for condensing multimaps.

**Theorem 2.7.** Let $X$ be a nonempty closed $\Gamma$-convex subset of a complete locally $G$-convex uniform space $(E; U, \Gamma)$ and $s : X \to X$ a surjection. If $T \in s$-$KKM(X, X, X)$ is closed, generalized condensing and satisfies that

$$T(A) \subseteq \overline{T(A)}$$

for any precompact subset $A$ of $X$, then $T$ has a fixed point.

**Proof.** By Lemma 2.6, there exists a precompact $\Gamma$-convex subset $K$ of $X$ such that $T(K) \subseteq K$, which in conjunction with $T(\overline{K}) \subseteq \overline{T(K)}$ shows that $T(\overline{K}) \subseteq \overline{K}$. Putting $D = s^{-}(\overline{K})$. Then $s : D \to \overline{K}$ is surjective and $T \in s$-$KKM(D, \overline{K}, \overline{K})$ is compact and closed. So, the conclusion follows from Theorem 1.4. 

**Theorem 2.8.** Let $X$ be a nonempty closed $\Gamma$-convex subset of a complete locally $G$-convex uniform space $(E; U, \Gamma)$ and $s : X \to X$ a surjection. If $T \in s$-$KKM(X, X, X)$ is l.s.c., closed and generalized condensing, then $T$ has a fixed point.

**Proof.** Since $T$ is l.s.c., we have that $T(A) \subseteq \overline{T(A)}$ for any subset $A$ of $X$, so the conclusion follows from Theorem 2.7.

When $X$ is a nonempty closed convex subset of a locally convex topological vector space, it is shown in [4] that every closed and generalized condensing self-multimap on $X$ with $s$-$KKM$ property has a fixed point. Hence, it is interesting to ask whether the condition of lower semicontinuity can be dropped
in Theorem 2.8.

**Theorem 2.9.** Let $X$ be a nonempty closed $\Gamma$-convex subset of a complete locally $G$-convex uniform space $(E; U, \Gamma)$ and $s : X \rightarrow X$ a surjection. Assume $T : X \rightarrow X$ satisfies

(2.9.1) $T$ is closed and generalized condensing;

(2.9.2) for any precompact subset $A$ of $X$, if

$$H(x) = T(x) \cap \overline{A} \neq \emptyset$$

for any $x \in \overline{A}$,

then $H \in s$-KKM($s^{-}(\overline{A}), \overline{A}, \overline{A}$).

Then $T$ has a fixed point.

**Proof.** Condition (2.9.1) and Lemma 2.6 show that there is a precompact $\Gamma$-convex subset $K$ of $X$ such that $T(K) \subseteq K$ and $T(x) \cap \overline{K} \neq \emptyset$ for any $x \in \overline{K}$. Define $H : \overline{K} \rightarrow \overline{K}$ by $H(x) = T(x) \cap \overline{K}$ for $x \in \overline{K}$. Then condition (2.9.2) gives us that $H \in s$-KKM($s^{-}(\overline{K}), \overline{K}, \overline{K}$). Obviously, $H$ is compact and closed. As $H$ satisfies all requirements of Theorem 1.4, it has a fixed point $\hat{x}$ which is also a fixed point of $T$. \hfill $\square$

If $T(\overline{K}) \subseteq \overline{T(K)}$ and $T \in s$-KKM($X, X, X$), then $T \in s$-KKM($s^{-}(\overline{K}), \overline{K}, \overline{K}$), so Theorem 2.7 can be derived from Theorem 2.9.

3. Applications

In this section, we shall deduce some quasi-equilibrium theorems as applications of the previous fixed point results.

**Theorem 3.1.** Let $X$ be a nonempty closed $\Gamma$-convex subset of a complete locally $G$-convex uniform space $(E; U, \Gamma)$, $s : X \rightarrow X$ a surjection, $f : X \times X \rightarrow \mathbb{R}$ an upper semicontinuous function and $H : X \rightarrow X$ a closed, condensing multimap with compact values. Assume the following two conditions hold:

(3.1.1) The function $M : X \rightarrow \mathbb{R}$ defined by

$$M(x) = \max_{y \in H(x)} f(x, y)$$

is lower semicontinuous.
(3.1.2.) The multimap $T : X \rightharpoonup X$ defined by

$$T(x) = \{y \in H(x) : f(x, y) = M(x)\}$$

is in $s$-KKM$(X, X, X)$ and satisfies $T(A) \subseteq \overline{T(A)}$ for any precompact subset $A$ of $X$.

Then there exists an $\hat{x} \in X$ such that $\hat{x} \in H(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$.

**Proof.** Since $f$ is u.s.c. and $H(x)$ is compact, $T(x)$ is nonempty for any $x \in X$. We claim that $T$ is closed. Let $(x_\alpha, y_\alpha)$ be a net in $Gr(T)$ and $(x_\alpha, y_\alpha) \to (x, y)$. We have

$$f(x, y) \geq \limsup_\alpha f(x_\alpha, y_\alpha) = \limsup_\alpha M(x_\alpha) \geq \liminf_\alpha M(x_\alpha) \geq M(x),$$

where the last inequality follows from (3.1.1). Since $H$ is closed and $y_\alpha \in H(x_\alpha)$ for any $\alpha$, we see that $(x, y) \in Gr(H)$. Hence $y \in \{z \in H(x) : f(x, z) = M(x)\}$, and so $T$ is closed. In addition, $T$ is condensing. In fact, if $A \subseteq X$ is not precompact, then, since $H$ is condensing, we have from Proposition 2.4 that

$$\Psi(A) \nsubseteq \Psi(H(A)) \subseteq \Psi(T(A)),$$

which shows that $T$ is condensing. Consequently, $T$ has a fixed point $\hat{x}$ by Theorem 2.7, that is, $\hat{x} \in H(\hat{x})$ and $f(\hat{x}, \hat{x}) = M(\hat{x})$. \hfill \Box

Suppose $\{(E_i; U_i, \Gamma_i)\}_{i \in I}$ is any family of locally $G$-convex uniform spaces. Let $E = \Pi_{i \in I} E_i$ be equipped with product topology induced by the product uniformity $U = \Pi_{i \in I} U_i$, and the product convexity structure $\Gamma = \Pi_{i \in I} \Gamma_i : \langle E \rangle \rightharpoonup E$ defined by

$$\Gamma(A) = \Pi_{i \in I} \Gamma_i(\pi_i(A))$$

for each $A \in \langle E \rangle$, where $\pi_i : E \to E_i$ is the projection of $E$ onto $E_i$. Then $(E; U, \Gamma)$ is a locally $G$-convex uniform space, cf. [5]. Moreover, if each $(E_i; U_i, \Gamma_i)$ is complete and if each $\Gamma_i$ has the property that $\Gamma_i$-co($A_i$) is precompact for any precompact subset $A_i$ of $E_i$, then $\Gamma$ has the similar property, that is, $\Gamma$-co($A$) is precompact whenever $A$ is precompact in $E$. To see this,
let $A$ be a precompact subset of $E$. Since each $\pi_i$ is uniformly continuous, we see that each $\pi_i(A)$ is precompact in $E_i$. Now, noting that

$$ A \subseteq \Pi_{i \in I} \Gamma_i \text{-co}(\pi_i(A)) \subseteq \Pi_{i \in I} \Gamma_i \text{-co}(\pi_i(A)) $$

and $\Pi_{i \in I} \Gamma_i \text{-co}(\pi_i(A))$ is $\Gamma$-convex, cf. [5, Lemma 2.5], we obtain that $\Gamma \text{-co}(A) \subseteq \Pi_{i \in I} \Gamma_i \text{-co}(\pi_i(A))$. So, the compactness of $\Pi_{i \in I} \Gamma_i \text{-co}(\pi_i(A))$ implies that $\Gamma \text{-co}(A)$ is precompact.

**Lemma 3.2.** Let $\{X_i\}_{i \in I}$ be a family of nonempty closed $\Gamma_i$-convex subsets, each in a complete locally $G$-convex uniform space $(E_i; U_i, \Gamma_i)$, $K_i$ a nonempty compact subset of $X_i$ and $T_i : X = \Pi_{i \in I} X_i \rightrightarrows K_i$ a l.s.c., closed multimap. Assume $T : X \rightrightarrows K = \Pi_{i \in I} K_i$ defined by $T(x) = \Pi_{i \in I} T_i(x)$ for each $x \in X$ has the KKM property. Then there is $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$, for any $i \in I$.

**Proof.** Obviously, $T$ is a compact and closed multimap. Furthermore, since each $T_i$ is l.s.c., so is $T$, cf. [7, Theorem 7.3.12]. Therefore, $T$ has a fixed point $\hat{x}$ by Theorem 2.8, that is, $\hat{x}_i \in T_i(\hat{x})$ for any $i \in I$. \hfill $\square$

**Theorem 3.3.** Let $\{X_i\}_{i \in I}$ be a family of nonempty closed $\Gamma_i$-convex subsets, each in a complete locally $G$-convex uniform space $(E_i; U_i, \Gamma_i)$, $K_i$ a nonempty compact subset of $X_i$, $H_i : X = \Pi_{i \in I} X_i \rightrightarrows K_i$ a closed multimap with compact values, and $f_i, g_i : X^{-i} \times X_i \to \mathbb{R}$ be upper semicontinuous functions, where $X^{-i}$ denotes $\Pi_{j \in I \setminus \{i\}} X_j$. Assume that

(3.3.1) $g_i(x) \leq f_i(x)$ for any $x \in X$;

(3.3.2) for any $i \in I$, the function $M_i : X = X^{-i} \times X_i \to \mathbb{R}$ defined by

$$ M_i(x) = \max_{y \in H_i(x)} g_i(x^{-i}, y) $$

is lower semicontinuous; and

(3.3.3) for any $i \in I$, the multimap $T_i : X \rightrightarrows K_i$ defined by

$$ T_i(x) = \{y \in H_i(x) : f_i(x^{-i}, y) \geq M_i(x)\} $$

is l.s.c.

(3.3.4) $T : X \rightrightarrows K = \Pi_{i \in I} K_i$ defined by $T(x) = \Pi_{i \in I} T_i(x)$ for each $x \in X$ has the KKM property. Then there exists an $\hat{x} \in X$ such that for each $i \in I$,

$$ \hat{x}_i \in H_i(\hat{x}) \text{ and } f_i(\hat{x}^{-i}, \hat{x}_i) \geq M_i(\hat{x}). $$
Proof. Firstly, note each $T_i(x)$ is nonempty by (3.3.1) since each $H_i(x)$ is compact and $g_i(x^{-i},\cdot)$ is u.s.c. on $H_i(x)$. Next, just as in the proof of Theorem 3.1, $Gr(T_i)$ is closed in $X \times K_i$ and $T_i$ is compact. Applying Lemma 3.2, we infer that there is an $\hat{x} \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for any $i \in I$, that is, $\hat{x}_i \in H_i(\hat{x})$ and $f_i(\hat{x}^{-i},\hat{x}_i) \geq M_i(\hat{x})$. □

References


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