Uniqueness of Positive Solutions for a Class of p-Laplacian Systems with Multiple Parameters

G. A. Afrouzi and E. Graily

Department of Mathematics, Faculty of Basic Sciences
Mazandaran University, Babolsar, Iran
afrouzi@umz.ac.ir

Abstract

We prove uniqueness of positive solution for the system

\[
\begin{align*}
\Delta_p u &= -\lambda f(v) \quad \text{in } \Omega, \\
\Delta_q v &= -\mu g(u) \quad \text{in } \Omega \\
u = v &= 0 \quad \text{on } \partial\Omega
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) and \(f(x) \sim x^p\), \(g(x) \sim x^q\) at \(\infty\) for some positive numbers \(p, q\) with \(p, q > 1\)

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Introduction

Consider the boundary value problem

\[
\begin{align*}
\Delta_p u &= -\lambda f(v); \quad x \in \Omega \\
\Delta_q v &= -\mu g(u); \quad x \in \Omega \\
u(x) &= v(x) = 0 \quad x \in \partial\Omega
\end{align*}
\]

where the p-Laplacian operator \(\Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z)\), \(\lambda, \mu\) are positive parameters and \(\Omega\) is bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial\Omega\) and \(f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+\).

In \([2,6]\) the authors consider the existence of positive solutions for the p-Laplacian system with large \(\lambda\):
\[
\begin{cases}
-\Delta_p u = \lambda f(v) & \text{in } \Omega \\
-\Delta_p v = \lambda g(u) & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{cases}
\]
(2)

\[
\begin{cases}
-\Delta u = \lambda f(v) & \text{in } \Omega \\
-\Delta v = \lambda g(u) & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{cases}
\]
(3)

Dalmasso in [3], discussed the system (3) when \( p = q = 2 \) and \( f, g \) are increasing and \( f, g \geq 0 \). In [2] Dalmasso proved existence and uniqueness of positive solutions to (1) when the composition \( f o (cg) \) is sublinear at \( \infty \) and superlinear at 0 for each \( c > 0 \). The arguments in [5] rely on the fact the positive solutions to (3) in a ball are radially symmetric and decreasing (see [4,10]). D.D.Hai in [1] discussed about uniqueness of positive solutions for semilinear elliptic of the system (3). In this paper we shall extend the results in [1],[5] to the case of a bounded domain in \( R^N \) and for a class of \( p \)-Laplacian systems. Results for the single equation case were obtained in [7,9]. Our approach is based on sub- and supersolution, and maximum principle, and weak comparison principle.

2. Existence and uniqueness results

We make the following assumptions

**\( \textbf{(H.1)} \)** \( f, g : R^+ \rightarrow R^+ \) are nondecreasing, continuous, \( C^1 \) on \( (0, \infty) \)
\[ \limsup_{x \to 0^+} x f(x) < \infty, \quad \limsup_{x \to 0^+} x g(x) < \infty \]

**\( \textbf{(H.2)} \)** There exist positive \( \beta, \delta, p, q \) with \( p, q > 1 \) such that
\[ \beta x^p \leq f(x) \leq \delta x^p, \quad \beta x^q \leq g(x) \leq \delta x^q \]
for all \( x \geq 0 \), and for \( p_1 > p, q_1 > q \),
\[ \frac{f(x)}{x^{p_1}} \text{ and } \frac{g(x)}{x^{q_1}} \text{ are nonincreasing for } x \text{ large.} \]

Our main result is

**Theorem 1.** Let \( \textbf{(H.1)} - \textbf{(H.2)} \) hold. Then system (1) has a unique positive solution for \( \min (\lambda^q - 1, \mu^p, \lambda^q \mu^{p-1}) \) large.

The next lemma provides estimates for solutions to (1). When \( \Omega \) is a ball, it was established in [5].
We shall denote the norm in $C^k(\Omega)$ by $|.|_k$.

**Lemma 1.** Let $(u, v)$ be a positive solution of (1). Then there exist positive $M$ and $M_i$, $1 \leq i \leq 4$, such that

$$M_1(\lambda q - 1 \mu p)^{1-(p+q)}d(x, \partial \Omega) \leq u(x) \leq M_2(\lambda q - 1 \mu p)^{1-(p+q)}d(x, \partial \Omega)$$

$$M_3(\lambda q - 1 \mu p)^{1-(p+q)}d(x, \partial \Omega) \leq v(x) \leq M_4(\lambda q - 1 \mu p)^{1-(p+q)}d(x, \partial \Omega)$$

for $\min (\lambda q - 1 \mu p, \lambda q - 1 \mu p) > M$. Here $d(x, \partial \Omega)$ denotes the distance for $x$ to $\partial \Omega$.

**Proof.** Let $(u, v)$ be a positive solution for (1). We first establish the upper estimate for $v$. In what follows, we shall denote by $C_i$ positive constant independent of $\lambda, \mu, u, v$. Using the equations for $u, v$, we obtain

$$u(x) = \lambda \int_{\Omega} K(x, y)f(v(y))dy, \quad v(x) = \mu \int_{\Omega} K(x, y)g(u(y))dy$$

where $K(x, y)$ denotes the Green’s function of $-\Delta_p(-\Delta_q)$ with Dirichlet boundary conditions. Thus, by (H.2),

$$-\Delta_p u = \lambda f(v) \leq \lambda \delta v^p$$

and with follow from weak comparison principle we have

$$|u|_0 \leq C_1(\lambda \delta |v|_{0}^{p})^{\frac{1}{p-1}}$$

and

$$-\Delta_q v = \mu g(u) \leq \lambda \delta u^q$$

and

$$|v|_0 \leq C_1(\mu \delta |u|_{0}^{q})^{\frac{1}{q-1}}$$

From (5), (7), it follow that

$$|u|_0 \leq C_2(\lambda q - 1 \mu p)^{\frac{1}{1-(p+q)}}$$

Which, we follow from [1] and with (H.2) and weak comparison principle and regularity estimates, implies

$$|v|_1 \leq C_3(\mu |g(u)|_0^q)^{\frac{1}{q-1}} \leq C_3(\mu \delta |u|_{0}^{q})^{\frac{1}{q-1}} \leq C_3(\mu \delta [C_2(\lambda q - 1 \mu p)^{\frac{1}{1-(p+q)}}]^{q})^{\frac{1}{q-1}} \equiv M_4(\lambda q - 1 \mu p)^{\frac{1}{1-(p+q)}}$$
and

\[ v(x) \leq M_4(\lambda^q \mu^{p-1})^{\frac{1}{(\frac{1}{p} + \frac{1}{q})}} d(x, \partial \Omega) \]

(9)

Follows from the mean value theorem. The upper estimate for \( u \) follows in the same manner.

Next, let \( x_0 \in \Omega \) and \( R > 0 \) be such that \( B \equiv B(x_0, R) \subset \Omega \). Here \( B(x_0, R) \) denotes the open ball centered at \( x_0 \) with radius \( R \). Then \( (u, v) \) is a supersolution for

\[
\begin{align*}
-\Delta_p u &= \lambda f(v) \quad \text{in } B \\
-\Delta_q v &= \lambda g(u) \quad \text{in } B \\
u = v = 0 \quad \text{on } \partial B
\end{align*}
\]

(10)

We shall construct a positive subsolution \( (u_0, v_0) \) for (10) with \( u_0 \leq u \) and \( v_0 \leq v \). To this end, let \( \epsilon > 0 \) and let \( \tilde{u}, \tilde{v} \) be the solution of

\[
\begin{align*}
-\Delta \tilde{u} &= \tilde{v}^p \quad \text{in } B \\
-\Delta \tilde{v} &= \tilde{u}^q \quad \text{in } B \\
\tilde{u} = \tilde{v} = 0 \quad \text{on } \partial B
\end{align*}
\]

(11)

whose existence follows from [3, 5]. Define \( u_0 = \epsilon \frac{1}{\tilde{u}} \tilde{u}, v_0 = \mu \beta \epsilon \frac{1}{\tilde{v}} \tilde{v} \), suppose that \( |\nabla u_0|^{p-2}, |\nabla v_0|^{q-2} \leq 1 \), \( \beta \) is given by (H.2). A direct calculation gives

\[
\Delta_p u_0 = \nabla \cdot (|\nabla u_0|^{p-2} \nabla u_0) = |\nabla u_0|^{p-2} \Delta u_0 = -|\nabla u_0|^{p-2}(\epsilon \frac{1}{\tilde{u}}) \tilde{v}^p \\
\geq -\lambda \beta (\mu \beta \epsilon \frac{1}{\tilde{v}} \tilde{v})^p \\
\geq -\lambda f(\beta \epsilon \frac{1}{\tilde{v}} \tilde{v}) \\
= -\lambda f(v_0)
\]

If \( \lambda^q \mu^p > 1 \) and \( \epsilon \) sufficiently small, and

\[
\Delta_q v_0 = |\nabla v_0|^{q-2} \Delta v_0 = -\mu \beta \epsilon \frac{1}{\tilde{v}} \tilde{u}^q = -\mu \beta (\epsilon \frac{1}{\tilde{u}}) \tilde{u}^q \\
\geq -\mu g(\epsilon \frac{1}{\tilde{u}} \tilde{u}) \\
= -\mu g(u_0),
\]

i.e., \( (u_0, v_0) \) is a subsolution for (10). Clearly \( u_0 \leq u \leq and v_0 \) in \( B \) for small \( \epsilon \). Hence there exists a solution \( (\overline{u}, \overline{v}) \) to (10) with \( \overline{u} \leq u, \overline{v} \leq v \). Since \( \overline{u} \) is radially symmetric, it follows from [5, Lemma 4] that
Uniqueness of positive solutions

\[ u(x) \geq \tilde{M}_1 (\lambda^{q-1} \mu^p)^{1-(p+q)} \quad \text{for} \ |x - x_0| \leq \frac{R}{2} \quad (12) \]

for \( \min(\lambda^{q-1} \mu^p, \lambda^q \mu^{p-1}) \) large, where \( \tilde{M}_1 \) is a positive constant independent of \( u, v, \lambda, \mu \).

Let \( \bar{\Omega} = \Omega \setminus B(x_0, \frac{R}{2}) \) and let \( \psi \) be the solution of

\[
\begin{align*}
\Delta_p \psi &= 0 \quad \text{in} \ \bar{\Omega}, \\
\psi &= 0 \quad \text{on} \ \partial \Omega, \\
\psi &= 1 \quad \text{on} \ \partial B(x_0, \frac{R}{2})
\end{align*}
\]

(13)

Since \( \Delta_p u \leq 0 \) in \( \Omega \), the maximum principle (see, e.g., [8, 10]) implies

\[ u(x) \geq (\tilde{M}_1 (\lambda^{q-1} \mu^p)^{\frac{1}{1-(p+q)})} \psi(x) \geq M_1 (\lambda^{q-1} \mu^p)^{\frac{1}{1-(p+q)}} \psi(x) \geq M_1 (\lambda^{q-1} \mu^p)^{\frac{1}{1-(p+q)}} d(x, \partial \Omega) \quad \text{in} \ \bar{\Omega}, \]

where \( \tilde{M}_1 \) is a positive constant satisfying \( \tilde{M}_1 \psi(x) \geq M_1 d(x, \partial \Omega) \) for \( x \in \bar{\Omega} \). Combine this and (12), we obtain the lower estimate for \( u \). This completes the proof of lemma 1. \( \square \)

**Lemma 2.** Let \( (u, v) \) be a solution to (1) and satisfy

\[
\begin{align*}
\Delta_p w_0 &= -g(u) \quad \text{in} \ \Omega, \\
w_0 &= 0 \quad \text{on} \ \partial \Omega,
\end{align*}
\]

(14)

Then for \( \min(\lambda^{q-1} \mu^p, \lambda^q \mu^{p-1}) \) large, there exist a positive number \( C \) independent of \( u, v, \lambda, \mu \), such that

\[ w_0(x) \geq C d(x, \partial \Omega) \quad \text{for} \ x \in \Omega. \]

**Proof.** Let \( \epsilon_0 > 0 \). It follows from (H.2) and lemma 1 that for \( (\lambda^{q-1} \mu^p, \lambda^q \mu^{p-1}) \) large,

\[ g(u(x)) \geq \beta (u(x))^q \geq \beta [M_1 (\lambda^{q-1} \mu^p)^{\frac{1}{1-(p+q)}}]^q > 1 \]

if \( d(x, \partial \Omega) > \epsilon_0 \). Thus

\[
\Delta_p w_0 \leq \begin{cases} -1 & \text{if} \ d(x, \partial \Omega) > \epsilon_0, \\
w_0 = 0 & \text{if} \ d(x, \partial \Omega) \leq \epsilon_0,
\end{cases}
\]

(15)
and the lemma follows from the maximum principle and weak comparison principle.

**Proof of theorem 1.** The existence part follows from [3]. Let \((u, v)\) and \((u_1, v_2)\) be solutions (1) and suppose that min \((\lambda^{q-1} \mu^p, \lambda^q \mu^{p-1})\) is large enough so lemma 1, 2 apply. By Lemma 1,

\[
\frac{M_1}{M_2} u_1 \leq u \leq \frac{M_2}{M_1} u_1 \quad \text{in } \Omega.
\]

Let \(\alpha = \sup\{c > 0 : u \geq cu_1 \text{ in } \Omega\}\). Then \(\alpha_0 \leq \alpha \leq \alpha_0^{-1}\), where \(\alpha_0 = \frac{M_1}{M_2}\). We claim that \(\alpha \geq 1\). Suppose to the contrary that \(\alpha < 1\). Let \(q_2 > q_1 > q, p_2 > p_1 > p\) and \(p, q > 1\). Let \(A > 0\) be such that

\[
\frac{g(x)}{x^{q_1}} \text{ is nonincreasing for } x > A
\]

and define \(\Omega_1 = \{x \in \Omega : u_1(x) > \frac{A}{\alpha_0}\}\). Then

\[
g(\alpha u_1(x)) \geq \alpha^{q_1} g(u_1(x)) \quad \text{for } x \in \Omega_1, \quad (16)
\]

While if \(x \in \Omega \setminus \Omega_1\),

\[
|g(u_1(x)) - g(\alpha u_1(x))| \leq K(1 - \alpha),
\]

where \(K = \frac{1}{\alpha_0} \sup\{|x \dot{g}(x)| : 0 < x \leq \frac{A}{\alpha_0}\}\), which implies

\[
g(u_1(x)) \geq g(\alpha u_1(x)) - K(1 - \alpha) \quad \text{for } x \in \Omega \setminus \Omega_1 \quad (17)
\]

Define the operator \(T : C(\overline{\Omega}) \rightarrow C(\overline{\Omega})\) by \(T(z) = w\) if

\[
\Delta_p w = -z \quad \text{in } \Omega_1, \quad w = 0 \quad \text{on } \partial \Omega_1.
\]

Let \(w = Tg(\alpha u_1)\). Then it follows from (16),(17) and maximum principle that \(w \geq \overline{w}\), where \(\overline{w}\) satisfies

\[
\Delta_p \overline{w} = \begin{cases} 
-\alpha^{q_1} g(u_1) & \text{in } \Omega_1, \\
-g(u_1(x)) + K(1 - \alpha) & \text{in } \Omega \setminus \Omega_1,
\end{cases} \quad \overline{w} = 0 \quad \text{on } \partial \Omega \quad (18)
\]

Let \(w_0 = Tg(u_1)\). Then \(\Delta_p w_0 = -g(u_1)\) and therefore

\[
\Delta_p (\overline{w} - \alpha^{q_1} w_0) = \begin{cases} 
0 & \text{in } \Omega_1, \\
(\alpha^{q_1} - 1) g(u_1) + K(1 - \alpha) & \text{in } \Omega \setminus \Omega_1, 
\end{cases}
\]

\[
\Delta_p (\overline{w} - \alpha^{q_1} w_0) = \begin{cases} 
0 & \text{in } \Omega_1, \\
(\alpha^{q_1} - 1) g(u_1) + K(1 - \alpha) & \text{in } \Omega \setminus \Omega_1, 
\end{cases}
\]

\[
\Delta_p (\overline{w} - \alpha^{q_1} w_0) = \begin{cases} 
0 & \text{in } \Omega_1, \\
(\alpha^{q_1} - 1) g(u_1) + K(1 - \alpha) & \text{in } \Omega \setminus \Omega_1, 
\end{cases}
\]
Note that there exists a positive constant $K_1$ depending only on $A, \alpha_0, K, q_1$, such that
\[
|\omega^n - 1| g(u_1) + K(1 - \alpha) | \leq K_1(1 - \alpha) \quad \text{in } \Omega \setminus \Omega_1
\]
we follow from [1] and weak comparison principle, we obtain,
\[
|\omega - \omega^n w_0|_1 \leq [K_1(1 - \alpha)]^{\frac{1}{p-1}} C.
\] (20)
Since
\[
M_1(\lambda^{q-1} \mu^p)^{\frac{1}{1-(p+q)}} d(x, \partial \Omega) \leq u_1(x) \leq \frac{A}{\alpha_0} \quad \text{on } \Omega \setminus \Omega_1
\]
it follows that
\[
d(x, \partial \Omega) \leq \frac{A}{\alpha_0 M_1(\lambda^{q-1} \mu^p)^{\frac{1}{1-(p+q)}}} \quad \text{for } x \in \Omega \setminus \Omega_1
\]
and therefore the right-hand side of (20) goes to 0 as $\lambda^{q-1} \mu^p \rightarrow \infty$. Let $\epsilon > 0$, then it follows from (20) and the mean value theorem and weak comparison principle that
\[
\omega(x) - \omega^n w_0(x) \geq -[\epsilon(1 - \alpha)]^{\frac{1}{p-1}} d(x, \partial \Omega), \quad x \in \Omega,
\]
for $\min(\lambda^{q-1} \mu^p, \lambda^q \mu^{p-1})$ large, which implies by Lemma 2 that
\[
\omega(x) - \omega^n w_0(x) \geq (\alpha^{q_1} - \alpha^{q_2}) w_0(x) - [\epsilon(1 - \alpha)]^{\frac{1}{p-1}} d(x, \partial \Omega)
\]
\[
\geq c\alpha^{q_1} (1 - \alpha^{q_2-q_1}) d(x, \partial \Omega) - [\epsilon(1 - \alpha)]^{\frac{1}{p-1}} d(x, \partial \Omega)
\]
\[
\geq [\min(1, q_2-q_1)c\alpha^{q_1} - \epsilon^{\frac{1}{p-1}} (1-\alpha)^{\frac{2-p}{p-1}}] (1-\alpha) d(x, \partial \Omega) > 0
\]
if $\epsilon$ sufficiently small. Consequently, $\omega(x) \geq \omega^n w_0(x) \geq \omega^{q_2} w_0(x)$, or
\[
Tg(\alpha u_1) \geq \alpha^{q_1} Tg(u_1).
\] (21)
Since
\[
\Delta q v = -\mu g(u) \leq -\mu g(\alpha u_1),
\]
it follows from (21) that
\[
v \geq (\mu Tg(\alpha u_1))^{\frac{1}{p-1}} \geq (\mu \alpha^{q_2} Tg(u_1))^{\frac{1}{p-1}} = \alpha^{q_2} v_1.
\]
This implies

\[ \Delta p u = -\lambda f(v) \leq -\lambda f(\alpha^q v_1). \]  \hspace{1cm} (22)

Now similarly

\[ T f(\alpha^q v_1) \geq \alpha^{pq_2} T f(v_1). \]  \hspace{1cm} (23)

(22),(23) and the maximum principle imply

\[ u \geq (\lambda T f(\alpha^q v_1))^{\frac{1}{p-1}} \geq (\lambda \alpha^{pq_2} T f(v_1))^{\frac{1}{p-1}} = \alpha^{pq_2} u_1, \]

Which is a contradiction since \( \alpha^{pq_2} > \alpha \). Thus \( \alpha \geq 1 \), i.e., \( u \geq u_1 \) and therefore \( u = u_1 \).

Similarly, \( v = v_1 \), completing the proof of Theorem 1. \( \square \)

References


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