Oscillation Theorems for Some Neutral Delay Differential Equations

Nina Shang, Fuyi Xu and Huizeng Qin

School of Mathematics and Information Science
Shandong University of Technology, Zibo, Shandong 255049, People’s Republic of China

Abstract

In this paper, we study the oscillation criteria for some differential equations with uncertain delay

\[ \left( r(t)\psi(y(t))|z'(t)|^{\gamma-1}z'(t) \right)' + q(t)f(x(g(t))) = 0, \]

where \( z(t) = x(t) + p(t)x(g_0(t)), y(t) = x(t) \) or \( y(t) = z(t) \). We point out some wrong results in the known paper. Moreover, by using inequality we obtain some extended Emden-Fowler differential equations

\[ \left( r(t)\psi(y(t))|z'(t)|^{\gamma-1}z'(t) \right)' + q_1(t)|x(g_1(t))|^\alpha - 1 x(g_1(t)) + q_2(t)|x(g_2(t))|^\beta - 1 x(g_2(t)) = 0. \]

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1 Introduction

We note that second-order ordinary differential equations are used in many fields such as vibrating masses attached to an elastic bar and some variational problems, see[3]. In recent years, there had been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of second order linear and nonlinear neutral delay differential equations.

1E-mail addresses: qinhz_000@163.com(H.Qin).
In [2], authors considered the second-order neutral delay differential equation

$$[x(t) + p(t)x(t - \tau)]'' + q(t)f(x(t - \sigma)) = 0.$$  \hspace{1cm} (1.1)

To the best of our knowledge, almost all of the known results obtained for (1.1) required the assumption that the function \( f(x) \) satisfies \( f'(x) \geq k > 0 \) or \( \frac{f(x)}{x} \geq k > 0 \) for \( x \neq 0 \) (see, [2] and the references therein). Recently, the results of papers [4, 5] for second-order ordinary differential equation were extended to (1.1) under the assumption that the nonlinear \( f(x) \) satisfies the sublinear condition

$$0 < \int_{0+}^{+\varepsilon} \frac{dx}{f(x)}, \int_{0-}^{-\varepsilon} \frac{dx}{f(x)} < \infty, \text{ for all } \varepsilon > 0,$$

as well as the superlinear condition

$$0 < \int_{0}^{\infty} \frac{dx}{f(x)} < \infty, \int_{-\varepsilon}^{-\infty} \frac{dx}{f(x)} < \infty, \text{ for all } \varepsilon > 0.$$

In recent years, Xu [1, 2] studied the following problem of oscillation of the Emden-Fowler neutral delay differential equation

$$\left(|z'(t)|^{-1} z'(t)\right)^{r} + q_1(t)|x(t-\sigma)|^{a-1} x(t-\sigma) + q_2(t)|x(t-\sigma)|^{\beta-1} x(t-\sigma) = 0, \hspace{1cm} (1.2)$$

where \( z(t) = x(t) + p(t)x(t - \tau) \). In what follows we assume that

(1) \( \tau \) and \( \sigma \) are nonnegative constants, \( \alpha, \beta \) and \( \gamma \) are positive constants with \( 0 < \alpha < \gamma < \beta \);

(2) \( q_1, q_2 \in C([t_0, \infty], R^+) \), where \( R^+ = (0, +\infty) \);

(3) \( p \in C([t_0, \infty], R) \) and \(-1 < p_0 \leq p(t) \leq 1 \), where \( p_0 \) is a constant.

Unfortunately, this statement about \(-1 < p_0 \leq p(t) \leq 1 \) is wrong. So some conclusions of [1] should be reconsidered.

Motivated by the works, our aim in this paper is to show oscillation criteria for some differential equations with uncertain delay

$$\left(r(t)\psi(y(t))|z'(t)|^{-1} z'(t)\right)^{r} + q(t)f(x(g(t))) = 0, \hspace{1cm} (1.3)$$

where \( z(t) = x(t) + p(t)x(g_0(t)), y(t) = x(t) \) or \( y(t) = z(t) \).

In this paper, we will use the following conditions

(A1) \( g(t), g_0(t) \leq t, \lim_{t \to \infty} g(t) = \lim_{t \to \infty} g_0(t) = \infty \), and \( g(t), g_0(t) \) are increasing;

(A2) \( r, q \in C([t_0, \infty), R^+) \), where \( R^+ = (0, \infty) \), \( R(t) = \int_{t_0}^{t} r^{-\frac{1}{r}}(s)ds, t_0 > 0, \lim_{t \to \infty} R(t) = \infty \);

(A3) \( p \in C([t_0, \infty), R) \) and \( 0 \leq p(t) \leq M_p \), where \( M_p \) is constant.

In addition, we make following assumptions for \( \psi(x), f(x) \)

(B1) \( \psi(x) \in C^1(-\infty, +\infty), 0 < \psi(x) \leq L^{-1}, \forall D > 0, \text{ there exist } k_f, K_f > 0 \) such that \( x f(x) \geq k_f |x|^\frac{1}{r}, |x| \geq D; \)
(B2) \( \psi(x) \in C^1(-\infty, +\infty) \), and \( \frac{\psi(z(t))}{\psi(x(t))} \leq K \) for \( 0 < \frac{z(t)}{x(t)} \leq M \). There exist \( k_f, K_f, k_\psi, K_\psi > 0 \) such that \( f(x) \geq k_f|x|^{\gamma_1}, \psi(x) \leq K_\psi|x|^{\gamma_1-\gamma}, \) \( |x| \geq D, \gamma_1 \geq \gamma \).

2 Main Results

By form invariance of equation (1.3), we may change \( x(t) \) into \(-x(t)\) in equation (1.3). Therefore, if solution \( x = x(t) \neq 0 \) of equation (1.3), we only discuss \( x = x(t) > 0 \). Instead, if solution \( x = x(t) \) of equation (1.3) is nonoscillation, then there exists \( T_1 > t_0 \) such that \( x(t) > 0 \) is increasing on \([T_1, \infty)\). By condition \( (A_1) \) we can know there exists \( T \geq T_1 \) such that \( g_0(t) \geq g(t) \geq T_1 \), for \( t \geq T \). So

\[
x(t), x(g_0(t)), x(g(t)) > 0, \text{ for } t \in [T, \infty).
\]

In this paper, we also assume that nonoscillation positive solution \( x = x(t) \) of equation (1.3) satisfies condition (2.1).

**Lemma 2.1.** Suppose that conditions \((A_1)-(A_3)\) and \( B_1 \) or \( B_2 \) hold, let \( x = x(t) \) is nonoscillation solution of equation (1.3). Then there exist \( T > t_0 \) such that

\[
z(t) > 0, z'(t) > 0, (r(t)\psi(y(t))z''(t))' < 0, \text{ for } t \geq T.
\]

**Proof.** Let \( x(t) \) be a nonoscillatory solution of equation (1.3), then \( z(t) > 0 \), for \( t \geq T \geq t_0 \). By equation (1.3), we have

\[
(r(t)\psi(y(t)))z''(t) = -q(t)f(x(g(t))) < 0.
\]

Next, we prove \( z'(t) > 0 \). If otherwise, we can know \( r(t)\psi(y(t))|z'(t)|^{\gamma_1}z'(t) < 0 \) and \( r(t)\psi(y(t))|z'(t)|^{\gamma_1}z'(t) \) is decreasing. So there exists \( T \geq t_0 \) such that

\[
r(t)\psi(y(t))|z'(t)|^{\gamma_1}z'(t) \leq r(\bar{T})\psi(y(\bar{T}))[z'(\bar{T})]^{\gamma_1}z'(\bar{T}), \text{ for } t \geq \bar{T}.
\]

We shall discuss it from two perspectives.

(i) If \( y(t) = z(t) \), then

\[-\psi^{\frac{1}{\gamma}}(z(t))z'(t) \geq Cr^{\frac{1}{\gamma}}(t), \text{ for } t \geq \bar{T}, \text{ where } C = (-r(\bar{T})\psi(y(\bar{T}))[z'(\bar{T})]^{\gamma_1}z'(\bar{T}))^{\frac{1}{\gamma}}.
\]

Integrating from \( \bar{T} \) to \( t \), we get

\[
\infty > -\int_{\bar{T}}^{t} \psi^{\frac{1}{\gamma}}(z)dz \geq -\int_{\bar{T}}^{z(t)} \psi^{\frac{1}{\gamma}}(z)dz \geq C(R(t) - R(\bar{T})) \to \infty (t \to \infty).
\]

(2.3)
Obviously, this is a contradiction.

(ii) If \( y(t) = x(t) \). One hand, if condition \((B_1)\) holds (or \( x = x(t) \) is decreasing ), let \( L^{-1} = \max_{x \in [0, x(T)]} \psi(x) \), we have

\[
z'(t) \leq -C_\gamma r^{1/\gamma}(t), \quad \text{where } C_\gamma = (-Lr(T)\psi(x(T))|z'(T)|^{\gamma-1}z'(T))^{1/\gamma}.
\]

Integrating from \( \tilde{T} \) to \( t \), we get

\[
z(t) \leq z(\tilde{T}) - C_\gamma(R(t) - R(\tilde{T})) \to \infty(t \to \infty),
\]

which contradicts to \( z(t) > 0 \).

On the other hand, if condition \((B_2)\) holds and \( x = x(t) \) is increasing, then

\[
1 \leq \frac{z(t)}{z(t)} = 1 + p(t)\frac{x(g_0(t))}{z(t)} \leq 1 + M_p, \quad \text{and}
\]

\[
-\psi^{1/\gamma}(z(t))z'(t) \geq Cr^{-1/\gamma}(t), \quad t \geq \tilde{T}, \quad \text{where } C = (-Kr(T)\psi(y(T))|z'(T)|^{\gamma-1}z'(T))^{1/\gamma}.
\]

By (2.3) we can know this is a contradiction.

We can easily get the following Lemma.

**Lemma 2.2.** If \( \gamma > 0 \), then

\[
at^\gamma - t^{\gamma+1} \leq \frac{\gamma^\gamma}{(\gamma + 1)^{\gamma+1}}a^{\gamma+1}, \quad t > 0.
\]
By (2.6), \( z'(g(t)) \geq z'(t) \) and derivating for (2.7), we have

\[
W'(t) \leq -\frac{k_f \mu(t)q(t)}{(1+\mu)p^\gamma} + \frac{\mu'(t)r(t)\psi(y(t))z'^\gamma(t)}{z'^\gamma(g(t))} - \frac{\gamma \mu(t)g'(t)r(t)\psi(y(t))z'^{\gamma+1}(t)}{z'^{\gamma+1}(g(t))}
\]

\[
\leq -\frac{k_f \mu(t)q(t)}{(1+\mu)p^\gamma} + \frac{\gamma}{L} \mu(t)g'(t)r(t) \left( \frac{\mu'(t)}{\gamma \mu(t)g'(t)s^\gamma} - s^{\gamma+1} \right) \text{ for } t \geq T,
\]

(2.8)

where \( s = \frac{z'(t)}{z(g(t))} \). By (2.4) we can know

\[
W'(t) \leq -\frac{k_f \mu(t)q(t)}{(1+\mu)p^\gamma} + \frac{r(t)\mu^{\gamma+1}(t)}{L(1+\gamma)^1}\mu^{\gamma+1}(t)g^\gamma(t).
\]

Integrating from \( t_0 \) to \( t \), we get

\[
\int_{t_0}^{t} \left( -\frac{k_f \mu(q(s))}{(1+\mu)p^\gamma} + \frac{r(s)\mu^{\gamma+1}(s)}{L(1+\gamma)^1}\mu^{\gamma+1}(s)g^\gamma(s) \right) \mu(s)ds \leq W(t_0) - W(t) < W(t),
\]

which contradicts to (2.5).

(ii) If \( x = x(t), x = x(g_i(t))(i = 0, 1, 2, 3) \) are decreasing on \((T, \infty)\), then \( z(t) \) is bounded and there exist numbers \( C_1, C_2 \) such that \( \lim_{t \to \infty} x(t) = C_1, \lim_{t \to \infty} z(t) = C_2 \). By \( z(t) \) is positive and increasing, we have \( C_1, C_2 > 0 \). So

\[
(r(t)\psi(y(t)))z'^\gamma(t)' \leq -q(t)f(C_1).
\]

By integrating and Lemma 2.1, we can know

\[
r(T)\psi(y(T))z'^\gamma(T) \geq r(T)\psi(y(T))z'^\gamma(T) - r(t)\psi(y(t))z'^\gamma(t) \geq f(C_1) \int_T^t q(s)ds,
\]

which contradicts to (2.5).

Therefore solution of equation (1.3) is oscillatory.

**Corollary 2.1.** Suppose conditions \((A_1) - (A_3)\) and \((B_1)\) hold, there exists nonnegative monotone function \( \mu(t) \in C^1(t_0, \infty) \) such that

\[
\lim \sup_{t \to \infty} \int_t^{t_0} q(s)ds = \infty, \quad \int_0^{\infty} \mu^{\gamma+1}(s)r(s) \frac{\mu^{\gamma+1}(s)g^\gamma(s)}{\mu^{\gamma+1}(s)g^\gamma(s)}ds < \infty.
\]

(2.9)

Then solution of equation (1.3) is oscillatory.

**Theorem 2.2.** Suppose conditions \((A_1) - (A_3)\) and \((B_2)\) hold, there exists a nonnegative monotone function \( \mu(t) \in C^1(t_0, \infty) \) such that

\[
\lim \sup_{t \to \infty} \int_0^{\infty} \left( \frac{k_f \mu(q(s))}{(1+\mu)p^\gamma} - \frac{\gamma K\psi r(g(s))\mu^{\gamma+1}(s)}{C^{\gamma+1}(\gamma+1)^\gamma \mu^{\gamma+1}(s)g^\gamma(s)} \right) \mu(s)ds = \infty,
\]

(2.10)
where
\[ C = \begin{cases} 1 + M_p, & y(t) = x(t), \\ 1, & y(t) = z(t). \end{cases} \]

Then solution of equation (1.3) is oscillatory.

**Proof**. The proof is similar to Theorem 2.1, we omit it.

**Corollary 2.2.** Suppose conditions $A_1 - A_3$ and $B_2$ hold, there exists non-negative monotone function $\mu(t) \in C^1(0, \infty)$ such that
\[ \limsup_{t \to \infty} \int_{t_0}^t q(s)\mu(s)ds = \infty, \quad \int_{t_0}^\infty \frac{\mu^{\gamma+1}(s)r(g(s))}{\mu^\gamma(s)g^\gamma(s)} < \infty. \tag{2.11} \]

Then solution of equation (1.3) is oscillatory.

## 3 Application

In this section, we give an example as an application to illustrate our results which extend and improve some known theorem in papers [1,2].

**Example 3.1.** Firstly, we consider the following equation which extends in paper [1,2]
\[ (r(t)\psi(y(t))|z'(t)|^{\gamma-1}z'(t))' + q_1(t)|x(g_1(t))|^{\alpha-1}x(g_1(t)) \]
\[ + q_2(t)|x(g_2(t))|^{\beta-1}x(g_2(t)) = 0, \tag{3.1} \]

where $z(t) = x(t) + p(t)x(g_0(t)), y(t) = x(t)$ or $y(t) = x(t), \alpha < \gamma < \beta$.

**Lemma 3.1.** If $x > 0, \alpha < \gamma < \beta$, then
\[ q_1(t)x^\alpha + q_2(t)x^\beta \geq q(t, \alpha, \beta, \gamma)x^\gamma, \tag{3.2} \]

where $q(t, \alpha, \beta, \gamma) = \frac{\beta - \alpha}{\beta - \gamma} \left( \frac{\gamma - \alpha}{\beta - \gamma} \right) \frac{\alpha + \gamma}{\beta + \gamma} \frac{2}{q_1^\alpha(t)q_2^\beta(t)}$.

Let $g(t) = \min\{g_1(t), g_2(t)\}$, by Lemma 3.1 and Equation (3.1) we have
\[ (r(t)\psi(y(t))|z'(t)|^{\gamma-1}z'(t))' + q(t, \alpha, \beta, \gamma)x^\gamma(g(t)) \leq 0. \tag{3.3} \]

Assume that the following conditions are satisfied:
(i) $r \in C[t_0, \infty), q(t) = q(t, \alpha, \beta, \gamma), p(t), g(t), g_0(t)$ satisfy condition $(A_1)$;
(ii) $\psi(x) \in C^1(-\infty, +\infty), 0 < \psi(x) \leq L^{-1}, \psi(x) \leq K_\psi|x|^{\gamma_1-\gamma}, |x| \geq D, \gamma_1 \geq \gamma$,

and
\[ \limsup_{t \to \infty} \int_{t_0}^t \left( q(t, \alpha, \beta, \gamma_1) - \frac{r(s)\mu^{\gamma+1}(s)}{L(\gamma + 1)^{\gamma+1}\mu^{\gamma+1}(s)g^\gamma(s)} \right) \mu(s)ds = \infty, \tag{3.4} \]

then solution of equation (3.1) is oscillatory.
Comparing Theorem 2.1 in paper [1], we give some remarks.

**Remark 3.1.** For condition (C₂) in paper [1], we consider Theorem 2.1 is wrong. For example, we choose \( p(t) = -\frac{1}{2}, \tau = 2, \sigma = 3, \gamma = 2, \alpha = 1, \beta = 3, r(t) = 1, \psi(x) = 1, \)
\[
q_1(t) = \frac{a^2(t - 3)^2a(3a - 2e^2)(t - 3) + 2(a - e^2)(2a - e^2)e^{-t+3}}{t - 3},
\]
\[
q_1(t) = -\frac{a^2e^{-g}}{t - 3}, a = \frac{e^2}{2} - 1
\]
satisfy condition (C₂) in paper [1]. But \( x(t) = te^{-t}(t \geq t_0 \gg 1) \) is nonoscillation positive solution of equation (1.2). So results in paper [1] is wrong.

**Remark 3.2.** Choose \( p(t) = 1, \) then \( Q_1(t) = 0, \) so condition (C₁) in paper [1] is not suitable. So Theorem 2.1 [1] of the application scope is limited. Our conditions (3.4) and (A₃) is better than condition (C₁)[1].

**Remark 3.3.** Function of condition (C₁) in paper [1] involves three functions \( \eta(t), \rho(t), h(t, s). \) But, our conditions (3.4) only involves a functions \( \mu(t). \) So our condition is simple.

**Example 3.2.** Consider the following equation
\[
((1 + rt^k)(1 + \varepsilon x^2(t))^{-1}|\eta'(t)|^{-\gamma}z'(t))' + q_1(t)|x((t - \sigma_1)^{k_1})(t - \sigma_1)^{k_1})
+ q_2(t)|x((t - \sigma_2)^{k_2})|^{-\beta}x((t - \sigma_2)^{k_2}) = 0,
\]
where \( z(t) = x(t) + [p_0 + p_1(2 + \sin t)t^\lambda]x((t - \sigma_0)^{k_0}), 0 < k_1, k_2 \leq k_0 \leq 1, 0 < \alpha < \beta, \gamma = \frac{\alpha + \beta}{2}, p_1, r \geq 0, \delta < \gamma, \lambda < 0, p_0 > 0, \varepsilon \geq 0. \)

Choose \( g(t) = (t - \sigma)^k, \sigma = \max\{\sigma_1, \sigma_2\}, k = \min\{k_1, k_2\}, \mu(t) = t^\mu, \mu \geq 0. \)

By \( \psi(y) = \frac{1}{1 + \varepsilon x^2(t)} \leq 1, \) we have
\[
q(s, \alpha, \beta, \frac{\alpha + \beta}{2}) = 2\sqrt{q_1(s)}q_2(s), \text{ for } t \geq \sigma_0, \sigma_1, \sigma_2,
\]
\[
\limsup_{t \to \infty} \int_{0}^{t} \frac{2\sqrt{q_1(s)}q_2(s)}{(1 + p_0 + p_1(2 + \sin s)(s - \sigma)^{\lambda k})^{\gamma}} - \frac{\mu^{\gamma+1}s^{-(\gamma+1)}(1 + rs^\delta)}{(\gamma + 1)^{\gamma+1}k^{\gamma}(s - \sigma)^{(k-1)}} s^\mu
\]
\[
= \infty
\]
Assume that the following conditions are satisfied:
(\text{i}) \( q_1(t)q_2(t) \geq K^2t^{2l}, \) let
\[
\mu = \begin{cases} 
-l - 1, l < -1, \\
0, l \geq -1,
\end{cases}
\]
if
\[
\begin{aligned}
&l > \gamma k - 1, \
&l > \delta - \gamma k - 1,
\end{aligned}
\]

(ii) \( q_1(t)q_2(t) \geq K^2t^2l \), let
\[
\mu = \begin{cases} 
-l - 1, & l < -1, \\
0, & l \geq -1,
\end{cases}
\]

if
\[
\begin{aligned}
l &< -\gamma k - 1, \\
l &< -\delta - \gamma k - 1,
\end{aligned}
\]

and
\[
\begin{aligned}
2k &
(1 + p_0 + 3p_1)\gamma
- \frac{\mu^{\gamma+1}}{(\gamma + 1)k^{\gamma}} > 0, \
2k &
(1 + p_0 + 3p_1)\gamma
- \frac{\mu^{\gamma+1}r}{(\gamma + 1)k^{\gamma}} > 0
\end{aligned}
\]

(iii) If \( \limsup_{t \to \infty} \int_{t_0}^{t} q(s, \alpha, \beta, \gamma) ds \), let \( \mu = 0 \).

By (3.6), (3.7) and condition (i)-(iii), we can guarantee (2.5) is suitable, so solution of equation (3.5) is oscillatory. Comparing Example 3.1-3.2 in paper[1], we give some remarks.

Remark 3.4. Choose \( p_1 = r = 0, k_0, k_1, k_2 = 1, 0 < \alpha < \beta, \gamma = \frac{\alpha + \beta}{2}, \sigma_0 = 1, \sigma_1 = \sigma_2 = 2, l > -\gamma - 1, q_1(t)q_2(t) \geq K^2t^2l \), so solution of equation (3.5) is oscillatory. In particular,

(i) \( l = -1, \varepsilon = 0 \), problem is Example 3.1 [1];

(ii) \( l = \gamma, \varepsilon = 0 \), problem is Example 3.2 [1];

(iii) \( l = -\gamma - 1, \varepsilon = 0 \), problem is Example 3.2 [2]. So our results extend some known results in paper[1, 2].

Remark 3.5. Choose \( p_1 = r = 0, k_0, k_1, k_2 = 1, 0 < \alpha < \beta, \gamma = \frac{\alpha + \beta}{2}, \sigma_0 = 1, \sigma_1 = \sigma_2 = 2, l = \gamma, \varepsilon = 0 \),

\[
q_1(t)q_2(t) = \begin{cases} 
\eta(t - 3n), & 3n < t < 3n + 1, \\
\eta(3n + 1 - t), & 3n + 1 < t < 3n + 2n > 0, n = 1, 2, \ldots, \\
q_0(t) \geq 0, & 3n + 2 < t < 3n + 3.
\end{cases}
\]

We suitably choose \( q_0(t) \) to guarantee \( q_1, q_2 \in C([t_0, \infty), R^+) \), so solution of equation (3.1) is oscillatory.
Example 3.3. Consider the following equation

\[
\left(1 + rt^h\right)|z(t)|^{\gamma-1}z'(t) + q_1(t)|x((t - \sigma_1)^{k_1}|^{\alpha-1}x((t - \sigma_1)^{k_1})
+ q_2(t)|x((t - \sigma_2)^{k_2}|^{\beta-1}x((t - \sigma_2)^{k_2}) = 0, \tag{3.8}
\]

where \(t \geq \sigma_0, \sigma_1, \sigma_2 \geq 0\), other parameters remain unchanged. Let \(\gamma = 1, \gamma_1 = \frac{\alpha + \beta}{2}, M_\varphi = 1, g(t) = (t - \sigma)^{k}, \sigma = \max\{\sigma_1, \sigma_2\}, k = \min\{k_1, k_2\}, \mu(t) = t^\mu, \mu \geq 0\). By Theorem 2.2, we can know solution of equation (3.8) is oscillatory.

References


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