On the Continuity of Restriction Maps

Yüksel Soykan

Zonguldak Karaelmas University
Art and Science Faculty
Department of Mathematics
67100, Zonguldak, Turkey
yuksel_soykan@hotmail.com

Abstract
In this paper, we prove that the restriction maps define continuous linear operators on the Hardy space of unit disk.

Mathematics Subject Classification. 30D55, 42B30, 58C07

Keywords: Hardy spaces, Restriction Maps, Continuity of Maps

1 Introduction

As usual, we define the Hardy space $H^2 = H^2(\Delta)$ as the space of all functions $f : z \to \sum_{n=0}^{\infty} a_n z^n$ for which the norm $\|f\|_{H^2} = \left( \sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}$ is finite. Here, $\Delta$ is the open unit disc. If $f \in H^2$ then we also have $\|f\|_{H^2}^2 = \frac{1}{2\pi} \int_{\partial \Delta} |f(z)|^2 |dz|$ where $\partial \Delta$ is the boundary of $\Delta$. The reader is referred to [1], [2], [3], and references therein for the basic properties of these spaces.

Let $\omega$ be an analytic and conformal map in a neighbourhood of $I = [0, 1]$ onto $\Delta$ and let $\gamma = \omega(I) \subseteq \overline{\Delta}$. If $S \subseteq \Delta$ is a simply connected domain, define the measure $\mu$ by

$$\mu(S) = \frac{\text{arc length measure of } (S \cap \gamma)}{2\pi}.$$ 

Note that

$$d\mu = \frac{|dz|}{2\pi} = \frac{1}{2\pi i} \frac{dz}{z}.$$ 

Definition 1 For $0 < h < 1$ and $\theta \in (0, 2\pi)$ let $E_{h\theta} = \{ re^{\theta} : 1 - h \leq r < 1; \theta_0 \leq \theta \leq \theta_0 + h \}$. A positive measure $\mu$ on $\overline{\Delta}$ is called a Carleson measure if there is some constant $A$ such that $\mu( E_{h\theta} ) \leq Ah$ for every $h$ and for every $\theta$. 
The main aim of this work is to prove the following theorem. For similar work regarding restriction maps, see [4].

**Theorem 1** Let \( \omega \) be analytic and conformal in a neighbourhood \( N \) of \( I = [0, 1] \) onto \( \Delta \) and let \( \gamma = \omega(I) \subseteq \overline{\Delta} \) (note that by definition then \( \gamma \) is an analytic arc). Then for every \( f \in H^2(\Delta) \), there is a constant \( C > 0 \) such that

\[
\int_{\gamma} |f(z)|^2 \frac{|dz|}{2\pi} \leq C^2 \|f\|_{H^2}^2
\]

(i.e. the restriction \( f \to f|_{\gamma} \) defines a continuous linear operator mapping \( H^2(\Delta) \) into \( L^2(\gamma, |dz|/2\pi) \)).

The proof is based on the Carleson theorem which we now state for the convenience. For a proof, see Duren [2].

**Theorem 2 (Carleson)** Suppose that \( \mu \) is a finite measure on \( |z| < 1 \). Then \( \mu \) is a Carleson measure if and only if there exists a constant \( C \) such that

\[
\int_{|z| < 1} |f(z)|^2 d\mu(z) \leq C^2 \|f\|_{H^2}^2 \quad \text{for all } f \in H^2.
\]

## 2 Proof of Theorem 1

Before giving the proof of Theorem 1 we need also two lemmas.

**Lemma 3** Let

\[
F(s, t) = \begin{cases} 
\frac{\omega(s) - \omega(t)}{s - t}, & \text{if } s \neq t \\
\omega'(t), & \text{if } s = t
\end{cases}
\]

Then there is some \( \delta > 0 \) such that \( |F(s, t)| \geq \delta \) on \( I \times I \) so that

\[
|\omega(s) - \omega(t)| \geq \delta |s - t|
\]

for \( s, t \in I \).

**Proof.** Let \( \Gamma \subset N \) be a curve. For \( s \neq t \)

\[
F(s, t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\zeta)}{s - t} (\frac{1}{\zeta - s} - \frac{1}{\zeta - t}) d\zeta = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(\zeta)}{(\zeta - s)(\zeta - t)} d\zeta
\]

also true for \( s = t \). So \( F \) is continuous on \( I \times I \) and nowhere 0 on \( I \times I \). Hence \( \inf_{s, t \in I} |F(s, t)| = \delta > 0 \).
Lemma 4 Let \( z_1 = r_1 e^{i\theta_1}, \ z_2 = r_2 e^{i\theta_2} \) where \( 0 \leq r_1, r_2 \leq 1 \) (i.e. \( r_1, r_2 \in \Delta \)) and \( \theta_1, \theta_2 \in \mathbb{R} \). Then
\[
|z_1 - z_2| \leq (r_1 - r_2)^2 + (\theta_1 - \theta_2)^2.
\]

Proof. We have
\[
|z_1 - z_2| = (r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2).
\]

Now
\[
\cos x \geq 1 - \frac{x^2}{2}
\]
for all \( x \in \mathbb{R} \). Hence
\[
|z_1 - z_2| \leq r_1^2 + r_2^2 - 2r_1r_2 + 2\frac{r_1r_2}{2}(\theta_1 - \theta_2)^2 
\leq (r_1 - r_2)^2 + (\theta_1 - \theta_2)^2.
\]

Proof of Theorem 1:
Note that if \( \gamma = \omega(I) \subseteq \partial \Delta \), then it is easy to see the validity of the required inequality. Now we consider the other cases of \( \gamma \) (i.e. \( \gamma \subseteq \overline{\Delta} \) but \( \gamma \nsubseteq \partial \Delta \)). Let \( \theta \in (0, 2\pi) \) and \( 0 < h < 1 \). Suppose \( \omega(t) \in E_{h\theta} \). For every \( s \in I \), we have \( \delta^2 |s - t|^2 \leq |\omega(s) - \omega(t)|^2 \). Suppose \( \omega(s) = r_1 e^{i\theta_1} \) and \( \omega(t) = r_1 e^{i\theta_2} \). Then
\[
(r_1 - r_2)^2 + (\theta_1 - \theta_2)^2 \geq \delta^2 |s - t|^2.
\]

So if \( |s - t|^2 \geq \frac{3h^2}{\delta^2} \), then
\[
(r_1 - r_2)^2 + (\theta_1 - \theta_2)^2 \geq 3h^2
\]
and so either
\[
|r_1 - r_2| > h \text{ or } |\theta_1 - \theta_2| > h
\]
for any argument \( \theta_1 \) for \( \omega(s) \), \( \theta_2 \) for \( \omega(t) \). In other case \( \omega(s) \notin E_{h\theta} \). So
\[
\omega^{-1}(E_{h\theta} \cap \gamma) \subseteq [t - \frac{\sqrt{3}}{\delta}h, \ t + \frac{\sqrt{3}}{\delta}h]
\]
and then
\[
\mu(E_{h\theta}) = \frac{\text{arc length measure of } (E_{h\theta} \cap \gamma)}{2\pi} = \frac{1}{2\pi} \int_{\omega^{-1}(E_{h\theta} \cap \gamma)} |\omega'(s)| \, ds
\leq \frac{1}{2\pi} \left( \frac{2\sqrt{3}}{\delta} \|\omega'\|_{\infty} \right) h = Ah.
\]
Now required result follows from Theorem 2.
References


Received: January 29, 2008