On Critical Exponent for Existence of Positive Solutions for Some Semipositone Problems Involving the Weight Function

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Abstract

In this paper, we study existence of positive solution for the semipositone problem of the form

\[-\Delta u = \lambda m(x)u^\alpha - c, \quad x \in \Omega,\]
\[u(x) = 0, \quad x \in \partial\Omega,\]

where \(\Delta\) denote the Laplacian operator, \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\) with \(\partial\Omega\) of class \(C^2\), \(\lambda, c\) are positive parameters and the weight \(m(x)\) satisfying \(m(x) \in C(\Omega)\) and \(m(x) \geq m_0 > 0\) for \(x \in \Omega\). A critical exponent is obtained for existence of positive solution by applying the method of sub-super solution.

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1 Introduction

In this work, we consider the existence of positive solution to boundary value problem of the form

\[-\Delta u = \lambda m(x)u^\alpha - c, \quad x \in \Omega,\]
\[u(x) = 0, \quad x \in \partial\Omega,\]  

(1)
where $\Delta$ denote the Laplacian operator, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ with $\partial \Omega$ of class $C^2$, $\lambda$, $c$ are positive parameters and the weight $m(x)$ satisfying $m(x) \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$.

Here we consider the challenging semipositone case $c > 0$. Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see [6, 7]). We refer to [1, 2, 3] for additional results in semipositone problems. Our approach is based on the method of sub-super solutions, see [4, 8].

## 2 Existence results

We first give the definition of sub-super solution of (1). A super solution to (1) is defined as a function $z \in C^2(\Omega)$ such that

$$-\Delta z \geq \lambda g(x, z) \quad x \in \Omega,$$

$$z \geq 0, \quad x \in \partial \Omega.$$

Sub solutions are defined similarly with the inequalities reversed and it is well known that if there exists a sub solution $\psi$ and a super solution $z$ to (1) such that $\psi(x) \leq z(x)$ for $x \in \bar{\Omega}$, then (1) has a solution $u$ such that $\psi(x) \leq u(x) \leq z(x)$ for $x \in \bar{\Omega}$. Further note that if $\psi(x) \geq 0$ for $x \in \Omega$ then $u \geq 0$ for $x \in \Omega$.

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases}
-\Delta \phi = \lambda \phi, & x \in \Omega, \\
\phi = 0, & x \in \partial \Omega.
\end{cases}$$

(2)

Let $\phi_1 \in C^1(\Omega)$ be the eigenfunction corresponding to the first eigenvalue $\lambda_1$ of (2) such that $\phi_1(x) > 0$ in $\Omega$, and $\|\phi_1\|_\infty = 1$. It can be shown that $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial \Omega$. Here $n$ is the outward normal. This result is well known (see [5]), and hence, depending on $\Omega$, there exist positive constants $k, \eta, \mu$ such that

$$\begin{align*}
\lambda_1 \phi_1^2 - |\nabla \phi_1|^2 & \leq -k, \quad x \in \bar{\Omega}_\eta, \\
\phi_1 & \geq \mu, \quad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\eta,
\end{align*}$$

(3)

with $\bar{\Omega}_\eta = \{x \in \Omega \mid d(x, \partial \Omega) \leq \eta\}$.

We will also consider the unique solution, $\zeta \in C^1(\bar{\Omega})$, of the boundary value problem

$$\begin{cases}
-\Delta \zeta = 1, & x \in \Omega, \\
\zeta = 0, & x \in \partial \Omega,
\end{cases}$$

(4)
to discuss our existence result. It is known that $\zeta > 0$ in $\Omega$ and $\frac{\partial \zeta}{\partial n} < 0$ on $\partial \Omega$.

Our main result is as follows:

**Theorem 2.1.** If $\alpha < 1$, then there exist positive constants $c_0 = c_0(\Omega)$ and $\lambda^* = \lambda^*(\Omega, c)$ such that (1) has a positive solution for $c \leq c_0$ and $\lambda \geq \lambda^*$.

**Proof.** To obtain the existence of positive solution to problem (1), we constructing a positive subsolution $\psi$ and supersolution $z$. We shall verify that $\psi = \frac{1}{2} \phi_1^2$ is a subsolution of (1). Since $\nabla \psi = \phi_1 \nabla \phi_1$, a calculation shows that

\[
-\Delta \psi = -\Delta \left( \frac{1}{2} \phi_1^2 \right) = -(|\nabla \phi_1|^2 + \phi_1 \Delta \phi_1) = [\phi_1 (-\Delta \phi_1) - |\nabla \phi_1|^2] = \lambda_1 \phi_1^2 - |\nabla \phi_1|^2.
\]

Then $\psi$ is a subsolution if

\[
\lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \leq \lambda m(x) \psi^\alpha - c.
\]

Now $\lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \leq -k$ in $\bar{\Omega}_\eta$, and therefore
if $c \leq c_0 = k$, then

\[
\lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \leq \lambda m(x) \psi^\alpha - c,
\]

Furthermore, we note that $\phi_1 \geq \mu > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$, also in $\Omega_0$ we have

\[
\lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \leq \lambda_1 \leq \lambda m(x) \psi^\alpha - c,
\]

if

\[
\lambda \geq \lambda^* = \frac{\lambda_1 + c}{\mu^\alpha m_0}.
\]

Hence if $c \leq c_0$ and $\lambda \geq \lambda^*$ then (3) is satisfy and $\psi$ is a subsolution.

Next, we construct a supersolution $z$ of (1). We denote $z = A \zeta(x)$, where the constant $A > 0$ is large and to be chosen later. We shall verify that $z$ is a
supersolution of (1). A calculation shows that
\[-\Delta z = A(-\Delta \zeta) = A.\]

Thus \(z\) is a supersolution if
\[A \geq \lambda m(x) z^\alpha - c,\]
and therefore if \(A \geq A_0\) where
\[A_0 = (\lambda \|m\|_\infty \|\xi\|_\infty \|\zeta\|_\infty) \frac{1}{1-\alpha},\]
we have
\[-\Delta z \geq \lambda m(x) z^\alpha - c,\]
and hence \(z\) is supersolution of (1). Since \(\zeta > 0\) and \(\partial \zeta/\partial n < 0\) on \(\partial \Omega\), we can choose \(A\) large enough so that \(\psi \leq z\) is also satisfied. Thus, by comparison principle, there exists a solution \(u\) of (1) with \(\psi \leq u \leq z\). This completes the proof of Theorem 2.1.

**References**


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