The Hadamard’s Inequality for $s$–Convex Function of 2-Variables on the Co-ordinates

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Abstract
In this paper the extension of Hadamard’s type inequality for $s$–convex function and $s$–convex functions on the co-ordinates defined in 2-variables and some applications are given.

Keywords: $s$–Hadamard’s inequality, $s$–Convex function, Jensen’s inequality

1 Introduction
Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$, with $a < b$. The following double inequality:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2} \quad (1)$$

is known in the literature as Hadamard’s inequality for convex mappings.

In [1] Hudzik and Maligrada considered among others the class of functions which are $s$–convex in the second sense. This class is defined in the following way: a function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be $s$–convex in the second sense if

$$f \left( \lambda x + (1 - \lambda) y \right) \leq \lambda^s f(x) + (1 - \lambda)^s f(y) \quad (2)$$

holds for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$. It can be easily seen that every 1–convex function is convex.
In [2] Dragomir and Fitzpatrick proved a variant of Hadamard’s inequality which holds for \( s \)-convex functions in the second sense.

**Theorem A**

Suppose that \( f : [0, \infty) \rightarrow [0, \infty) \) is an \( s \)-convex function in the second sense, where \( s \in (0, 1) \) and let \( a, b \in [0, \infty), \ a < b \). If \( f \in L^1 [0, 1] \), then the following inequalities hold:

\[
2^{s-1} f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{s+1} \tag{3}
\]

the constant \( k = \frac{1}{s+1} \) is the best possible in the second inequality in (1.3). The above inequalities are sharp.

In [3], Dragomir established the following similar inequality of Hadamard-type for co-ordinated convex mapping on a rectangle from the plane \( \mathbb{R}^2 \).

Precisely, if \( f : [a, b] \times [c, d] \rightarrow \mathbb{R} \) is convex function one can define the following mapping on \( [0, 1]^2 \) such as:

\[
H(t, r) = \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f \left( tx + (1-t) \frac{a+b}{2}, ry + (1-r) \frac{c+d}{2} \right) \, dy \, dx
\]

**Theorem B**

Suppose that \( f : \Delta \rightarrow \mathbb{R} \) is co-ordinated convex on \( \Delta \). Then one has the inequalities

\[
f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \tag{4}
\]

The above inequalities are sharp.

Also, one can has the following properties for \( H \) (see [3-5]):

(i) \( H \) is co-ordinated convex on \( [0, 1]^2 \).

(ii) One has the bounds

\[
\sup_{(t, r) \in [0,1]^2} H(t, r) = \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dx \, dy = H(1, 1)
\]
and
\[ \inf_{(t,r)\in[0,1]^2} H(t,r) = f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) = H(0,0). \]

## 2 Hadamard’s Inequality

First of all, let us start with the following definition:

**Definition 2.1**

Consider the bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( [0, \infty)^2 \) with \( a < b \) and \( c < d \). The mapping \( f : \Delta \rightarrow \mathbb{R} \) is \( s \)-convex on \( \Delta \) if
\[
 f (\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda^s f (x, y) + (1 - \lambda)^s f (z, w)
\]
holds for all \((x, y), (z, w) \in \Delta \) with \( \lambda \in [0, 1] \) and for some fixed \( s \in (0, 1] \).

A function \( f : \Delta \rightarrow \mathbb{R} \) is \( s \)-convex on \( \Delta \) is called co-ordinated \( s \)-convex on \( \Delta \) if the partial mappings \( f_y : [a, b] \rightarrow \mathbb{R}, f_y (u) = f (u, y) \) and \( f_x : [c, d] \rightarrow \mathbb{R}, f_x (v) = f (x, v) \), are \( s \)-convex for all \( y \in [c, d] \) and \( x \in [a, b] \) with some fixed \( s \in (0, 1] \).

**Lemma 2.1**

Every \( s \)-convex mapping \( f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty) \) is \( s \)-convex on the co-ordinates, but the converse is not true in general.

**Proof.**

Suppose that \( f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty) \) is \( s \)-convex on \( \Delta \). Consider the function \( f_x : [c, d] \rightarrow [0, \infty), f_x (v) = f (x, v) \). Then for \( \lambda \in [0, 1] \) and \( v_1, v_2 \in [c, d] \), one has:
\[
 f_x (\lambda v_1 + (1 - \lambda) v_2) = f (x, \lambda v_1 + (1 - \lambda) v_2) \\
 = f (\lambda x + (1 - \lambda) x, \lambda v_1 + (1 - \lambda) v_2) \\
 \leq \lambda^s f (x, v_1) + (1 - \lambda)^s f (x, v_2) \\
 = \lambda^s f_x (x, v_1) + (1 - \lambda)^s f_x (x, v_2). 
\]

Therefore, \( f_x (v) = f (x, v) \) is \( s \)-convex on \([c, d] \).
The fact that \( f : [a, b] \rightarrow [0, \infty) \), \( f_y(u) = f(u, y) \) is also \( s \)-convex on \([a, b] \) for all \( y \in [c, d] \) goes likewise and we shall omit the details.

In [3] Dragomir gave a mapping \( f_0 : [0, 1]^2 \rightarrow [0, \infty) \) defined by \( f_0(x, y) = xy \) which is convex on the co–ordinates but is not convex. We consider the same function with \( s = 1 \) to prove that the \( s \)-convexity on the co–ordinates does not imply the \( s \)-convexity.

The following inequality is considered the mapping connected with the inequality (3).

**Theorem 2.1**

Suppose that \( f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow [0, \infty) \) is \( s \)-convex function on the co–ordinates on \( \Delta \). Then one has the inequalities:

\[
 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
 \leq 2^{s-2} \left[ \frac{1}{b-a} \int_a^b f \left( x, \frac{c + d}{2} \right) dx + \frac{1}{d-c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy \right] \\
 \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
 \leq \frac{1}{2(s+1)} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \\
 + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\
 \leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{(s+1)^2}.
\]
Proof.

Since \( f : \Delta \to \mathbb{R} \) is co-ordinated \( s \)-convex on \( \Delta \) it follows that the mapping \( g_x : [c, d] \to [0, \infty), g_x(y) = f(x, y) \) is \( s \)-convex on \([c, d]\) for all \( x \in [a, b] \). Then by \( s \)-Hadamard’s inequality (3) one has:

\[
2^{s-1} g_x \left( \frac{c+d}{2} \right) \leq \frac{1}{d-c} \int_c^d g_x(y) \, dy \leq \frac{g_x(c) + g_x(d)}{s+1}, \quad \forall x \in [a, b].
\]

That is,

\[
2^{s-1} f \left( x, \frac{c+d}{2} \right) \leq \frac{1}{d-c} \int_c^d f(x, y) \, dy \leq \frac{f(x, c) + f(x, d)}{s+1}, \quad \forall x \in [a, b].
\]

Integrating this inequality on \([a, b]\), we have

\[
\frac{2^{s-1}}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \leq \frac{1}{s+1} \left[ \frac{1}{b-a} \int_a^b f(x, c) \, dx + \frac{1}{b-a} \int_a^b f(x, d) \, dx \right].
\]

A similar arguments applied for the mapping \( g_y : [a, b] \to [0, \infty), g_y(x) = f(x, y) \), we get

\[
\frac{2^{s-1}}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy \leq \frac{1}{(d-c)(b-a)} \int_c^d \int_a^b f(x, y) \, dx \, dy \leq \frac{1}{s+1} \left[ \frac{1}{d-c} \int_c^d f(a, y) \, dy + \frac{1}{d-c} \int_c^d f(b, y) \, dy \right].
\]

Summing the inequalities (6) and (7), we get the second and the third inequalities in (5).

Therefore, by \( s \)-Hadamard’s inequality (3), we also have:

\[
f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{2^{s-1}}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy
\]

and

\[
f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{2^{s-1}}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx
\]
which give, by addition the first inequality in (5).

Finally, by the same inequality we can also state:

\[
\frac{1}{b-a} \int_{a}^{b} f(x, c) \, dx \leq \frac{f(a, c) + f(b, c)}{s + 1}
\]

\[
\frac{1}{b-a} \int_{a}^{b} f(x, d) \, dx \leq \frac{f(a, d) + f(b, d)}{s + 1}
\]

\[
\frac{1}{d-c} \int_{c}^{d} f(a, y) \, dy \leq \frac{f(a, c) + f(a, d)}{s + 1}
\]

and

\[
\frac{1}{d-c} \int_{c}^{d} f(b, y) \, dy \leq \frac{f(b, c) + f(b, d)}{s + 1}
\]

which give, by addition the last inequality in (5).

**Note 1:** In (5) if \( s = 1 \) then the inequality reduced to inequality (4).

Now, for the mapping \( H \) we have the following result(s):

**Theorem 2.2**

Suppose that \( f : \Delta \to \mathbb{R} \) is co-ordinated \( s \)-convex on \( \Delta \). Then for \( H(t, r) \) one has the inequalities

(i) \( H \) is co-ordinated \( s \)-convex on \([0, 1]^2\).

(ii) The mapping \( H \) is monotonic nondecreasing on the co-ordinates.

(iii) One has the bounds

\[
\inf_{(t,r) \in [0,1]^2} H(t,r) = H(0,0),
\]

and

\[
\sup_{(t,r) \in [0,1]^2} H(t,r) = H(1,1)
\]
Proof.

(i) Fix $r \in [0,1]$. Then for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0,1]$, we have:

\[
H(\alpha t_1 + \beta t_2, r) = \frac{1}{(b-a)(d-c)} \\
\times \int_a^b \int_c^d f \left( (\alpha t_1 + \beta t_2) x + (1 - (\alpha t_1 + \beta t_2)) \frac{a+b}{2}, ry + (1 - r) \frac{c+d}{2} \right) dy dx
\]

\[
= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f \left( t_1 x + (1 - t_1) \frac{a+b}{2} \right) \\
+ \beta \left( t_2 x + (1 - t_2) \frac{a+b}{2} \right), ry + (1 - r) \frac{c+d}{2} \right) dy dx
\]

\[
= \alpha^s \cdot \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f \left( t_1 x + (1 - t_1) \frac{a+b}{2}, ry + (1 - r) \frac{c+d}{2} \right) dy dx
\]

\[
+ \beta^s \cdot \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f \left( t_2 x + (1 - t_2) \frac{a+b}{2}, ry + (1 - r) \frac{c+d}{2} \right) dy dx
\]

\[
= \alpha^s H(t_1, r) + \beta^s H(t_2, r).
\]

Similarly, if $t \in [0,1]$ is fixed, then for all $r_1, r_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we also have:

\[
H(t, \alpha r_1 + \beta r_2) \leq \alpha^s H(t, r_1) + \beta^s H(t, r_2)
\]

and the statement is proved.

(ii) Firstly, we will show that

\[
H(t, r) \geq H(0, r), \forall t, r \in [0,1]^2.
\] (10)

By $s$–Hadamard’s inequality (3), we have

\[
H(t, r) \geq \frac{1}{d-c} \int_c^d f \left( \frac{1}{b-a} \int_a^b \left[ tx + (1 - t) \frac{a+b}{2} \right] dx, ry + (1 - r) \frac{c+d}{2} \right) dy
\]

\[
= \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, ry + (1 - r) \frac{c+d}{2} \right) dy = H(0, r).
\]

$\forall t, r \in [0,1]^2$. 

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Now, let $0 \leq t_1 < t_2 \leq 1$. By the $s$–convexity of the mapping $H (\circ, r)$ for all $r \in [0, 1]$, we have
\[
\frac{H (t_2, r) - H (t_1, r)}{t_2 - t_1} \geq \frac{H (t_1, r) - H (0, r)}{t_1} \geq 0.
\]
Note that, for the last inequality we have used (10).

(iii) Since $f$ is $s$–convex on the co–ordinates, we have by Jensen’s inequlaity for integrals that:
\[
H (t, r) = \frac{1}{b - a} \int_a^b \left[ \frac{1}{d - c} \int_c^d f \left( tx + (1 - t) \frac{a + b}{2}, ry + (1 - r) \frac{c + d}{2} \right) dy \right] dx
\]
\[
\geq \frac{1}{b - a} \int_a^b f \left( tx + (1 - t) \frac{a + b}{2} \right) \frac{1}{d - c} \int_c^d \left[ ry + (1 - r) \frac{c + d}{2} \right] dy \right) dx
\]
\[
= \frac{1}{b - a} \int_a^b f \left( tx + (1 - t) \frac{a + b}{2}, c + d \right) dx
\]
\[
\geq f \left( \frac{1}{b - a} \int_a^b \left[ tx + (1 - t) \frac{a + b}{2} \right] dx, c + d \right)
\]
\[
= f \left( \frac{a + b}{2}, c + d \right)
\]
\[
= H (0, 0).
\]
By the $s$–convexity of $H$ on the co–ordinates, we have
\[
H (t, r) \leq r \cdot \frac{1}{d - c} \int_c^d \left[ t \cdot \frac{1}{b - a} \int_a^b f (x, y) dy \right] dx
\]
\[
+ (1 - t) \frac{1}{b - a} \int_a^b f \left( \frac{a + b}{2}, y \right) dx \right] dy
\]
\[
+ (1 - t) \frac{1}{d - c} \int_c^d \left[ t \cdot \frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) dx
\]
\[
+ (1 - t) f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right] dy
\]
\[
\begin{align*}
= rt \cdot \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx + r (1-t) \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \, dy \\
+ t (1-r) \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) \, dx + (1-r) (1-t) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)
\end{align*}
\]

Therefore, by (6), (7), (8) and (9) we deduce that
\[
H(t, r) \leq \left[ rt + r (1-t) + t (1-r) + (1-t) (1-r) \right]
\times \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx
\]
\[
= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx
\]
\[
= H(1, 1).
\]

Thus, the second bound in (iii) is proved.

**Note 2:** In Theorem 2.2 set \( s = 1 \) we get the result obtained in Theorem B.

**Corollary 2.1**

Suppose that \( f : \Delta \to \mathbb{R} \) is co-ordinated \( s \)-convex on \( \Delta \). Define the mapping \( h : [0, 1] \to \mathbb{R}, h(t) = H(t, t) \). Then \( h \) is convex monotonic nondecreasing on \([0, 1]\) and one has the bounds:
\[
\inf_{t \in [0, 1]} h(t) = h(0) = H(0, 0),
\]
and
\[
\sup_{t \in [0, 1]} h(t) = h(1) = H(1, 1).
\]

**Proof.**

It’s an immediate consequence of Theorem 2.2.
Comment(s):

In the next paper we will give the most main applications on these result(s) by obtaining some Hadamard–type inequality for co–ordinated $s$–convex functions in a rectangle from the plane.

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References


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