Interval Oscillation Criteria Related to Interval Averaging Technique for Certain Nonlinear Delay Differential Equations

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Abstract

We present new interval oscillation criteria for certain nonlinear delay second order differential equation

\[(r(t)x'(t))' + q(t)f(x(\tau(t)))g(x'(t)) = 0\]

that are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of \([t_0, \infty)\), rather than on the whole half-line. Our results extend and improve some previous oscillation criteria and handle the cases which are not covered by known results and implied that the delay \(\tau(t)\) does not effect the oscillation. In particular, several examples that dwell upon the sharp condition of our results are also included.

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1 Introduction

We consider the interval oscillation behavior of solutions of the second order nonlinear delay differential equation

\[(r(t)x'(t))' + q(t)f(x(\tau(t)))g(x'(t)) = 0,\]

on the half-line \([t_0, \infty)\). In equation (1), we assume that \(q(t)\) is a positive continue function on \([t_0, \infty)\), \(r > 0\) is an eventually positive function, \(\tau(t)\)
is a positive continuously differentiable function on $[t_0, \infty)$ such that $\tau(t) \leq t$, $\tau'(t) > 0$, $t \geq t_0$, $\lim_{t \to \infty} \tau(t) = \infty$. $f \in C((\infty, +\infty), (-\infty, +\infty))$, $g \in C((\infty, +\infty); [0, \infty))$, $f(x) \geq K > 0$, $g(x) \geq C > 0$ for $x \neq 0$, where $K$ and $C$ are constants.

We recall that a function $x : [t_0, t_1) \to (-\infty, +\infty)$, $t_1 \geq t_0$ is called a solution of Eq.(1) if $x(t)$ satisfies Eq.(1) for all $t \in [t_0, t_1)$. In the sequel, it will be always assumed that solutions of Eq.(1) exist for any $t_0 \geq 0$. A solution $x(t)$ of Eq.(1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Finally, Eq.(1) is called oscillatory if all its proper solutions are oscillatory.

Eq.(1) with $\tau(t) \equiv t$ and $q(t) \geq 0$ has been studied by Grace and Lalli[13]. They mentioned that, though stability, boundedness and convergence to zero of all solutions of Eq.(1) have been investigated in the papers of Burton and Grimmer [14], Graef and Spikes[18,19], Lalli[20], and Wong and Burton[15]. Nothing much has been known regarding the oscillatory behavior of Eq.(1) except for the result by Wong and Burton[15, Th4] regarding the oscillatory behavior of the equation

$$x''(t) + q(t)f(x(t-\tau))g(x'(t)) = 0, \quad \tau > 0 \tag{2}$$

in connection with that of the corresponding linear equation

$$x''(t) + q(t)x(t) = 0. \tag{3}$$

Recently when assuming that $q(t) \geq 0$ for $t \in [t_0, \infty)$, Rogovchenko[3] presented new oscillation criteria which ensure the oscillatory character of Eq.(2). They are different from those of Grace and Lalli[13] and are applicable to other classes of equations which are not covered by the results of Grace and Lalli[13]. However, all the above mentioned oscillation results involve the integral of $q$ and hence require the information of $q$ on the entire half-line $[t_0, \infty)$, what’s more, $q(t) \geq 0$ for $t \in [t_0, \infty)$ is also required.

From the Sturm Separation Theorem, we see that oscillation is only an interval property, i.e. if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \to \infty$, such that for each $i$ there exists a solution of Eq.(3) that has at least two zeros in $[a_i, b_i]$, then every solution of Eq.(3) is oscillatory. El-Sayed[7] established an interval criterion for oscillation of a forced second order equation, but the result is very sharp, because a comparison with equations of constant coefficient is used in the proof. In 1997, Huang[8] and A. Elbert presented some interval criteria for oscillation and non-oscillation of Eq.(3). In 2000, Wan-Tong Li and Ravi P. Agarwal[2] obtained several interval criteria for Eq.(1) with $\tau(t) \equiv t$.

In [8], Mashfound considered the Euler differential equation with constant
delay of the form \( \tau(t) = t - \tau \), that is
\[
x''(t) + \frac{\lambda}{t^2} x(t - \tau) = 0, \quad t \geq t_0 > \tau > 0, \tag{4}
\]
and obtained that the delay does not effect the oscillation. In other words, this equation oscillates if and only if the corresponding equation without delay
\[
x''(t) + \frac{\lambda}{t^2} x(t) = 0, \quad t \geq t_0
\]
oscillates, that is, if and only if \( \lambda > 1/4 \). In 1995, Li and Yeh [6] further proved that the delay \( \tau(t) = t + \tau \) in Eq.(4) does not effect the oscillation, where \( \tau \) is a positive constant. The other related results can refer to Dzurina and Stavroulakis [10].

Motivated by the idea of El-Sayed [7], Kong [1], Li and Agarwal [2,12], by using averaging functions and a generalized Riccati technique, in this paper we obtain several new interval criteria for oscillation of Eq.(1), that is, criteria given by the behavior of Eq.(1) only on a sequence of subinterval of \( [t_0, \infty) \).

In the sequel we say that a function \( H = H(t, s) \) belongs to a function class \( X \), denoted by \( H \in X \) if \( H \in C(D, [0, \infty)) \), where \( D = \{(t, s), -\infty < s \leq t < \infty\} \) which satisfies
\[
H(t, t) = 0, H(t, s) > 0, \quad t > s,
\]
and has partial derivatives \( \partial H/\partial t \) and \( \partial H/\partial s \) on \( D \) such that
\[
\frac{\partial H}{\partial t} = h_1(t, s)\sqrt{H(t, s)}, \quad \frac{\partial H}{\partial s} = -h_2(t, s)\sqrt{H(t, s)}
\]
where \( h_1, h_2 \) are nonnegative continuous functions on \( D \).

2 Main Results

**Theorem1** Assume that there exists a positive, nondecreasing function \( \rho(t) \in C^1([t_0, \infty)) \) such that for some \( H \in X \) and for each sufficiently large \( T_0 \geq t_0 \), there exist increasing divergent sequence of positive numbers \( \{a_n\}, \{b_n\}, \{c_n\} \) with \( T_0 \leq a_n < c_n < b_n \) such that
\[
\frac{1}{H(c_n, a_n)} \int_{a_n}^{c_n} H(s, a_n) K C \rho(s) q(s) ds + \frac{1}{H(b_n, c_n)} \int_{c_n}^{b_n} H(b_n, s) K C \rho(s) q(s) ds
\]
\[
> \frac{1}{4H(c_n, a_n)} \int_{a_n}^{c_n} \frac{r(\tau(s)) \rho(s) \left[ h_2(b_n, s) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(b_n, s)} \right]^2}{\tau'(s)} ds
\]
\[
+ \frac{1}{4H(b_n, c_n)} \int_{c_n}^{b_n} \frac{r(\tau(s)) \rho(s) \left[ h_1(s, a_n) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(s, a_n)} \right]^2}{\tau'(s)} ds.
\]
Then every solution of Eq.(1) is oscillatory.

**Proof:** Suppose to the contrary. Then without loss of generality we may assume that there is a solution \( x(t) \) of Eq.(1) such that \( x(t) > 0 \) and \( x(\tau(t)) > 0 \) for \( t \geq T_1 \geq t_0 \) because a similar analysis holds for \( x(t) < 0 \) and \( x(\tau(t)) < 0 \). Then by (1), we obtain that

\[
(r(t)x'(t))' \leq 0
\]  

for \( t \geq T \geq \max\{T_0, T_1\} \). Define

\[
u(t) = \rho(t) \frac{r(t)x'(t)}{x(\tau(t))}.
\]

Then \( u(t) \) satisfies the Riccati equation

\[
u'(t) + \frac{\rho(t)q(t)f(x(\tau(t)))g(x'(t))}{x(\tau(t))} + \frac{u(t)x'(\tau(t))\tau'(t)}{x(\tau(t))} - \frac{\rho'(t)}{\rho(t)}u(t) = 0.
\]

By (5) and the assumptions of Theorem 1, we conclude that

\[r(\tau(t))x'(\tau(t)) \geq r(t)x'(t),\]

consequently, we obtain

\[KC\rho(t)q(t) \leq -u'(t) - \frac{u^2(t)\tau'(t)}{r(\tau(t))\rho(t)} + \frac{\rho'(t)}{\rho(t)}u(t).
\]

Multiplying (6) by \( H(t, s) \), integrating it with respect to \( s \) from \( c_n \) to \( t \) for \( t \in [c_n, b_n] \), we get that

\[
\int_{c_n}^{t} H(t, s)KC\rho(s)q(s)ds \\
\leq -\int_{c_n}^{t} H(t, s)u'(s)ds - \int_{c_n}^{t} H(t, s)\frac{u^2(s)\tau'(s)}{r(\tau(s))\rho(s)}ds + \int_{c_n}^{t} H(t, s)\frac{\rho'(s)}{\rho(s)}u(s)ds \\
= H(t, c_n)u(c_n) - \int_{c_n}^{t} h_2(t, s)\sqrt{H(t, s)u(s)ds} - \int_{c_n}^{t} H(t, s)\frac{u^2(s)\tau'(s)}{r(\tau(s))\rho(s)}ds \\
+ \int_{c_n}^{t} H(t, s)\frac{\rho'(s)}{\rho(s)}u(s)ds \\
\leq H(t, c_n)u(c_n) + \int_{c_n}^{t} \left[ h_2(t, s)\sqrt{H(t, s) + H(t, s)\frac{\rho'(s)}{\rho(s)}} \right] u(s)ds \\
- \int_{c_n}^{t} H(t, s)\frac{u^2(s)\tau'(s)}{r(\tau(s))\rho(s)}ds.
\]  

Note that

\[
\left[ h_2(t, s)\sqrt{H(t, s) + \frac{\rho'(s)}{\rho(s)}H(t, s)} \right] u(s) - H(t, s)\frac{u(s)\tau'(s)}{r(\tau(s))\rho(s)}
\]
\[ \leq \frac{r(\tau(s))\rho(s)}{4\tau'(s)} \left( h_2(t, s) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(t, s)} \right)^2, \]

hence, (7) implies
\[ f_{c_n}^{t} H(t, s) KC \rho(s) q(s) ds \]
\[ \leq H(t, c_n) u(c_n) + \frac{1}{4} \int_{c_n}^{t} r(\tau(s))\rho(s) \left( h_2(t, s) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(t, s)} \right)^2 ds. \]

Letting \( t \to b_n^- \) in the above, we obtain
\[ \frac{1}{H(b_n, c_n)} \int_{c_n}^{b_n} H(b_n, s) KC \rho(s) q(s) ds \]
\[ \leq u(c_n) + \frac{1}{4H(b_n, c_n)} \int_{c_n}^{b_n} r(\tau(s))\rho(s) \left( h_2(t, s) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(t, s)} \right)^2 ds. \] (8)

Next we go back to (6). We multiply (6) by \( H(s, t) \), integrate it with respect to \( s \) from \( t \) to \( c_n \) for \( t \in (a_n, c_n) \), we get that
\[
\begin{align*}
& f_{c_n}^{t} H(s, t) KC \rho(s) q(s) ds \\
& \leq - f_{c_n}^{t} H(s, t) u'(s) ds - \int_{c_n}^{t} H(s, t) \frac{u^2(s)r'(s)}{r(\tau(s))\rho(s)} ds + \int_{t}^{c_n} H(s, t) \frac{\rho'(s)}{\rho(s)} u(s) ds \\
& = - H(c_n, t) u(c_n) + \int_{c_n}^{t} H(s, t) \sqrt{H(s, t)} u(s) ds - \int_{t}^{c_n} H(s, t) \frac{u^2(s)r'(s)}{r(\tau(s))\rho(s)} ds \\
& + \int_{c_n}^{t} H(s, t) \frac{\rho'(s)}{\rho(s)} u(s) ds \\
& = - H(c_n, t) u(c_n) + \int_{c_n}^{t} \left[ h_1(s, t) \sqrt{H(s, t)} + H(s, t) \frac{\rho'(s)}{\rho(s)} \right] u(s) ds \\
& - \int_{c_n}^{t} H(s, t) \frac{u^2(s)r'(s)}{r(\tau(s))\rho(s)} ds. \quad (9)
\end{align*}
\]

Note that
\[
\left[ h_1(s, t) \sqrt{H(s, t)} + H(s, t) \frac{\rho'(s)}{\rho(s)} \right] u(s) - H(s, t) \frac{u^2(s)r'(s)}{r(\tau(s))\rho(s)} \\
\leq \frac{r(\tau(s))\rho(s)}{4\tau'(s)} \left[ h_1(s, t) \sqrt{H(s, t)} + H(s, t) \frac{\rho'(s)}{\rho(s)} \right]^2,
\]

hence (9) implies
\[
\begin{align*}
& f_{t}^{c_n} H(s, t) KC \rho(s) q(s) ds \\
& \leq - H(c_n, t) u(c_n) + \int_{t}^{c_n} \frac{r(\tau(s))\rho(s)}{4\tau'(s)} \left[ h_1(s, t) \sqrt{H(s, t)} + H(s, t) \frac{\rho'(s)}{\rho(s)} \right]^2 ds.
\end{align*}
\]
Letting $t \to a^+_n$ in the above, we obtain

$$f_{a_n}^{c_n} H(s, a_n) KC \rho(s) q(s) ds$$

$$\leq -H(c_n, a_n) u(c_n) + f_{a_n}^{c_n} \frac{r(\tau(s)) \rho(s)}{\tau'(s)} \left[ h_1(s, t) \sqrt{H(s, t) + H(s, a_n) \frac{\rho'(s)}{\rho(s)}} \right]^2 ds. \tag{10}$$

Then divide both sides in (10) by $H(c_n, a_n)$ to get

$$\frac{1}{H(c_n, a_n)} f_{a_n}^{c_n} H(s, a_n) KC \rho(s) q(s) ds$$

$$\leq -u(c_n) + \frac{1}{4H(c_n, a_n)} f_{a_n}^{c_n} \frac{r(\tau(s)) \rho(s)}{\tau'(s)} \left[ h_1(s, a_n) \sqrt{H(s, a_n)} + H(s, a_n) \frac{\rho'(s)}{\rho(s)} \right]^2 ds. \tag{11}$$

Now we claim that every solution of Eq.(1) has at least one zero $t_n > a_n$ for $a_n > T$. This follows since $x(t) > 0$ and $x(\tau(t)) > 0$ for $t > T$. Adding (8) and (11), we have the inequality

$$\frac{1}{H(c_n, a_n)} f_{a_n}^{c_n} H(s, a_n) KC \rho(s) q(s) ds + \frac{1}{H(b_n, c_n)} f_{c_n}^{b_n} H(b_n, s) KC \rho(s) q(s) ds$$

$$\leq \frac{1}{4H(c_n, a_n)} f_{a_n}^{c_n} \frac{r(\tau(s)) \rho(s)}{\tau'(s)} \left[ h_2(b_n, s) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(s, b_n)} \right]^2 ds$$

$$+ \frac{1}{4H(b_n, c_n)} f_{c_n}^{b_n} \frac{r(\tau(s)) \rho(s)}{\tau'(s)} \left[ h_1(s, a_n) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(s, a_n)} \right]^2 ds,$$

which contradicts the assumption. Thus, the claim holds, i.e., no nontrivial solution of Eq.(1) can be eventually positive. Therefore, every solution of Eq.(1) is oscillatory. The proof is complete.

**Theorem 2** Assume that for each sufficiently large $T \geq t_0$ there exist $a, b \in R$ such that $T < b < a$, and a positive, nondecreasing function $\rho(t) \in C^1([t_0, \infty))$ such that for any $\Phi \in C[a, b]$ satisfying $\Phi'(t) \in L^2[a, b]$ and $\Phi(a) = \Phi(b) = 0$, we have

$$\int_a^b \left[ \Phi^2(s) KC \rho(s) q(s) - \frac{r(\tau(s)) \rho(s)}{\tau'(s)} \left( \Phi'(s) + \Phi(s) \frac{\rho'(s)}{\rho(s)} \right) \right]^2 ds > 0.$$ 

Then every solution of Eq.(1) is oscillatory.

**Proof:** Without loss of generality, we may assume that $x(t) > 0$, and $x(\tau(t)) > 0$ for $t \geq T_1 \geq t_0$, because a similar analysis holds for $x(t) < 0$,
and \(x(\tau(t)) < 0\). Similar to the proof of Theorem 1, we multiply (6) by \(\Phi^2(t)\), integrate it with respect to \(s\) from \(a\) to \(b\) and use \(\Phi(a) = \Phi(b) = 0\), then we get

\[
\int_a^b \Phi^2(t)KCq(s)\rho(s)ds \\
\leq \int_a^b \Phi^2(t)u'(s)ds - \int_a^b \Phi^2(t)u^2(s)\frac{\tau'(s)}{r(\tau(s))}\rho(s)ds + \int_a^b \rho'(s)u(s)\Phi^2(s)ds \\
= 2 \int_a^b \Phi(s)\Phi'(s)u(s)ds - \int_a^b \Phi^2(t)u^2(s)\frac{\tau'(s)}{r(\tau(s))}\rho(s)ds + \int_a^b \rho'(s)u(s)\Phi^2(s)ds \\
= - \int_a^b \left[ \frac{\tau'(s)}{r(\tau(s))}\Phi(s)u(s) - \sqrt{\frac{r(\tau(s))\rho(s)}{\tau'(s)}} \left( \Phi'(s) + \Phi(s)\frac{\rho'(s)}{\rho(s)} \right) \right]^2 ds \\
+ \int_a^b \frac{r(\tau(s))\rho(s)}{\tau'(s)} \left( \Phi'(s) + \Phi(s)\frac{\rho'(s)}{\rho(s)} \right)^2 ds \\
\leq 0.
\]

Which contradicts the assumption. So every solution of Eq.(1) is oscillatory. The proof is complete.

**Corollary 1** If there exists a positive, nondecreasing function \(\rho(t) \in C^1([t_0, \infty))\), such that

\[
\limsup_{t \to \infty} \int_l^t \left[ H(s, l)KC\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4\tau'(s)} \left( h_1(s, l) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(s, l)} \right)^2 \right] ds > 0
\]

and

\[
\limsup_{t \to \infty} \int_l^t \left[ H(t, s)KC\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4\tau'(s)} \left( h_2(t, s) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(t, s)} \right)^2 \right] ds > 0
\]

for some \(H \in X\) and for each \(l \geq t_0\), then every solution of Eq.(1) is oscillatory.

**Proof:** For any \(T \geq t_0\), let \(a_n = T\). In (12), we choose \(l = a_n\), then there exists \(c_n > a_n\) such that

\[
\int_{a_n}^{c_n} \left[ H(s, a_n)KC\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4\tau'(s)} \left( h_1(s, a_n) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(s, a_n)} \right)^2 \right] ds > 0.
\]
In (13) we choose $l = c_n$, then there exists $b_n > c_n$ such that

$$\int_{c_n}^{b_n} \left[ H(b_n, s)KC\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4\tau'(s)} \left( \frac{h_2(b_n, s) + \frac{\rho'(s)}{\rho(s)} \sqrt{H(b_n, s)}}{4\tau'(s)} \right)^2 \right] ds > 0.$$  (15)

Combing (14) and (15), the conclusion thus comes from Theorem 1. The proof is complete.

### 3 Example

In this section we will show the applications of our oscillation criteria by an example. We will see that the equation in the example is oscillatory based on the results in Section 2.

**Example 1** Consider the nonlinear differential equation

$$x''(t) + t^2x\left(\frac{1}{2}t\right)(1 + x^2(t))^2(1 + (x'(t))^4) = 0, \quad t \geq 1.$$  

Here, $K = C = 1, r(t) \equiv 1, q(t) = t^2$. Let $\rho(t) \equiv 1, \Phi(t) = \sin t$. In fact, for any $T \geq 0$, there exist $k \in \mathbb{N}$, such that $2k\pi > T$. Let $a = 2k\pi, b = 2k\pi + \pi$, since

$$\int_{2k\pi}^{(2k+1)\pi} (s^2 \cdot \sin^2 t - \cos t) dt = \frac{1}{6}\pi^3 - \frac{1}{4}\pi > 0,$$

thus, the equation is oscillatory by Theorem 2.

### References


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