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On Annihilator of an Element of Finsler Module

A. Taghavi

Department of Mathematics
University of Mazandaran, Babolsar, Iran
taghavi@nit.ac.ir

Abstract

In this paper, we consider $\mathcal{A}$-valued seminorms on modules over a C^*-algebra $\mathcal{A}$ that has the formal properties of a seminorm. We show that if E be a left A-module over a C^*-algebra $\mathcal{A}$ and $\rho : E \to A_+$ be an $\mathcal{A}$-valued Finsler seminorm then for every $0 \neq x \in E$, $I_x = \{a \in \mathcal{A} : \quad \rho(ax) = 0\}$ is a closed two sided ideal of $\mathcal{A}$ and also if $\rho(x) \in Z(\mathcal{A})$ then quotient algebra $\mathcal{A}/I_x$ is commutative, where $Z(\mathcal{A})$ is center of $\mathcal{A}$.

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1 Introduction

Finsler modules over C^*-algebras are generalization of Hilbert C^*- modules which is studied in [1] and [6]. It is a useful tool in operator theory and operator algebras, and may be served as a noncommutative version of Banach bundles which are a main concept in Finsler geometry.

Definition 1.1 *Let $\mathcal{A}_+$ be the positive cone of a C^*-algebra $\mathcal{A}$ and E is a complex linear space which is a left $\mathcal{A}$-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in C, a \in \mathcal{A}$ and $x \in E$). An $\mathcal{A}$-valued Finsler seminorm is a map $\rho : E \to \mathcal{A}_+$ such that*

1) the map $\rho_E : x \to ||\rho(x)||$ is a seminorm on E, and

2) $\rho(ax)^2 = a\rho(x)^2a^*$ for each $a \in \mathcal{A}$ and $x \in E$.

An $\mathcal{A}$-valued Finsler seminorm is $\mathcal{A}$-valued Finsler norm if it satisfies $x = 0$ if $\rho(x) = 0$. E is equipped with a $\mathcal{A}$-valued Finsler norm is called a pre-Finsler $\mathcal{A}$-module. If $(E,\ \rho_E = ||.||_E)$ is complete then E is called a Finsler $\mathcal{A}$-module.

If we use the convention $|b| = (bb^*)^{1/2}$ for $b \in \mathcal{A}$, the condition (2) is equivalent to

$$\rho(ax) = |a\rho(x)|.$$

For $\mathcal{A}$ commutative this is the same as $\rho(ax) = |a|\rho(x)$, which is the usual form this sort of axiom takes in the commutative case. But this last version is not appropriate in the noncommutative case.

In this note we get some results about Finsler $\mathcal{A}$-modules. We show that $\mathcal{A}$-valued seminorms on modules over a C^*-algebra $\mathcal{A}$ has the formal properties of a seminorm. We prove that if ρ be an $\mathcal{A}$-valued seminorm on E and $x \in E$ then $I_x = \{a \in \mathcal{A} : \quad \rho(ax) = 0\}$ is a closed left ideal of $\mathcal{A}$. It is called annihilator (or order) of x. In addition, If E be an A-module over a C^*-algebra $\mathcal{A}$ and $\rho : E \to A_+$ be an $\mathcal{A}$-valued Finsler seminorm then for every $0 \neq x \in E$ which $\rho(x) \in Z(\mathcal{A})$, I_x is a closed two sided ideal and quotient algebra $\mathcal{A}/I_x$ is commutative, where $Z(\mathcal{A})$ is center of $\mathcal{A}$.

2 preliminaris

Lemma 2.1 *Let $\mathcal{A}$ be a C^*-algebra and ρ be an $\mathcal{A}$-valued Finsler seminorm on left A-module E then $\rho_E(ax)| \leq ||a||\rho_E(x)$ for all $a \in \mathcal{A}$ and $x \in E$.*

Proof. The condition (2) definition (1.1) implies that

$$\rho_E(ax) = ||\rho(ax)|| = |||a\rho(x)||| = ||a\rho(x)|| \leq ||a||||\rho(x)|| \leq ||a||\rho_E(x).$$

Proposition 2.2 *Let $\mathcal{A}$ be a C^*-algebra and ρ be an $\mathcal{A}$-valued Finsler seminorm on left A-module E. Then,*
1) $\rho(0) = 0$,
2)$N_\rho = \{x : \quad \rho(x) = 0\}$ is a submodule of E.

Proof. Statement (1) follows from $\rho(ax) = |a\rho(x)|$, with $a = 0$.

2) If $\rho(x) = \rho(y) = 0$ and $a \in \mathcal{A}$, by using Lemma (1.1) and subadditivity of ρ_E we have

$$0 \leq \rho_E(ax + y) \leq \rho_E(ax) + \rho_E(y) \leq ||a||\rho_E(x) + \rho_E(y) = 0.$$

This proves (2).

Theorem 2.3 *2.3. Let E be a left $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$ and ρ be a $\mathcal{A}$-valued Finsler seminorm on E. Then quotient $\mathcal{A}$-module E/N_ρ is a pre-Finsler $\mathcal{A}$-module with the module structure $a(x + N_\rho) = ax + N_\rho$ and $\rho^{'}(x + N_\rho) = \rho(x)$.*

proof. Recall that N_ρ is a submodule of E . First we show that $\rho^{'}$ is well-defined. Let $x, y \in [x]$ then $x - y \in N_\rho$. Suppose that $\rho(x) \neq \rho(y)$. Then $||\rho(x)^2 - \rho(y)^2|| \neq 0$ and so [6 Theorem 4] implies that there exist $a \in \mathcal{A}$ with $||a|| \leq 1$ such that $||a\rho(x)^2a|| \neq ||a\rho(y)^2a||$. Hence $\rho_E(ax)^2 = ||\rho(ax)^2|| \neq ||\rho(ay)^2|| = \rho_E(ay)^2$. In the other hand $ax - ay \in N_\rho$ for all $a \in A$ because by Proposition 2.2 N_ρ is a submodule of E. Thus, $0 \leq |\rho_E(ax) - \rho_E(ay)| \leq \rho_E(ax - ay) = 0$, so that $\rho_E(ax) = \rho_E(ay)$. This is a contradiction.

Now we must show that $\rho^{'}$ satisfies in two condition definition pre-Finsler $\mathcal{A}$-module. We have

$$\rho^{'}(a(x + N_\rho)) = \rho(ax) = |a\rho(x)| = |a\rho^{'}(x + N_\rho)|.$$

Now we show that $||x + N_\rho||_E = ||\rho^{'}(x + N_\rho)|| = ||\rho(x)||$ is a norm on E/N_ρ. $||x + N_\rho||_E = 0$ if and only if $||\rho(x)|| = 0$ if an only if $x \in N_\rho$ if and only if $x + N_\rho = N_\rho$.

$$||x + y + N_\rho||_E = ||\rho(x + y)|| \leq ||\rho(x)|| + ||\rho(y)|| = ||x + N_\rho||_E + ||y + N_\rho||_E.$$

And also for every $\lambda \in C$

$$||\lambda(x + N_\rho)||_E = ||\rho(\lambda x)|| = |||\lambda\rho(x)||| = |\lambda|||\rho(x)||.$$

So that $||\lambda(x + N_\rho)||_E = |\lambda|||x + N_\rho||_E$. These prove that $||.||_E$ is a norm.

Proposition 2.4 *Let $\mathcal{A}$ be a commutative C^*-algebra and ρ be an $\mathcal{A}$-valued Finsler seminorm on left A-module E. Then ρ is subadditive and for all $x, y \in E$ we have,*

$$|\rho(x) - \rho(y)| \leq \rho(x - y).$$

Proof. By Theorem 2.3 quotient $\mathcal{A}$-module E/N_ρ is a pre-Finsler $\mathcal{A}$-module with the module structure $a(x + N_\rho) = ax + N_\rho$ and $\rho^{'}(x + N_\rho) = \rho(x)$. By [6 Proposition 2] ρ' is subadditive. Hence, $\rho(x + y) = \rho^{'}(x + y + N_\rho) \leq \rho^{'}(x + N_\rho) + \rho^{'}(x + N_\rho) = \rho(x) + \rho(y)$. So that ρ is subadditive. Therefore,

$$\rho(x) = \rho(x - y + y) \leq \rho(x - y) + \rho(y).$$

So that $\rho(x) - \rho(y) \leq \rho(x - y)$. Thus it follows that

$$\begin{aligned} |\rho(x) - \rho(y)|^2 &= (\rho(x) - \rho(y))(\rho(x) - \rho(y))^* \\ &= (\rho(x) - \rho(y))(\rho(x) - \rho(y)) \\ &\leq \rho(x - y)\rho(x - y) = \rho(x - y)^2. \end{aligned}$$

So that by [5 Theorem 2.2.6] $|\rho(x) - \rho(y)| \leq \rho(x - y)$.

If ρ is an $\mathcal{A}$-norm on $\mathcal{A}$-module E where $\mathcal{A}$ is a C*-algebra, then clearly proposition (2.4) follows that ρ is continuous with norm induce of ρ by [5 Theorem 2.2.5].

3 Main Results

Theorem 3.1 *Let E be a left $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$ and ρ be an $\mathcal{A}$-valued Finsler seminorm on E. Then $I_x = \{a \in \mathcal{A} : \quad \rho(ax) = 0\}$ is a closed left ideal of $\mathcal{A}$.*

proof. To prove that I_x is a left ideal is clear.

Let $b \in \overline{I_x}$. There exist $b_n \in I_x$ such that $b_n \to b$. Therefore by Lemma 2.1 $b_n x + N_\rho \to bx + N_\rho$. By [6 corollary 5] ρ' which be defined in the as Theorem 2.3 is continuous. Thus $0 = \rho'(b_n x + N_\rho) \to \rho'(bx + N_\rho)$. Then $\rho(bx) = \rho'(bx + N_\rho) = 0$. Therefore $b \in I_x$. Hence I_x is a closed left ideal.

In Theorem 3.3, we prove that if $\rho(x) \in Z(\mathcal{A})$ then I_x is two-sided ideal.

It was proved in [2, P. 46], Lemma 2 that every seminorm ρ on a Banach algebra $\mathcal{A}$ with condition $\rho(x) \leq r(x),\ \forall x \in \mathcal{A}$ satisfies in $\rho(xy - yx) = 0,\ \ \forall x, y \in \mathcal{A}$. In the sequel we shall need the following analogue mentioned the Lemma.

Lemma 3.2 *Let $\mathcal{A}$ be a C^*-algebra and E be a left $\mathcal{A}$-module and ρ be an $\mathcal{A}$-valued seminorm which satisfies in condition $||\rho(ax)|| \leq r(a)||\rho(x)||,\ \forall a \in \mathcal{A}$ for some $x \in E$. Then $\rho((ab - ba)x) = 0, \quad \forall a, \forall b \in \mathcal{A}$.*

Proof. First Suppose that ρ is an $\mathcal{A}$-valued norm. Fix $b \in \mathcal{A}$ and, define

$$f(z) = e^{za} b e^{-za} x, \quad \forall z \in \mathcal{C}.$$

In view of Lemma 2.1 f is a entire function on $\mathbf{C}$. We have

$$f(z) = bx + z(ab - ba)x + ...$$

$$\begin{aligned} \rho_E(bx + z(ab - ba)x + ...) &= ||\rho(e^{za} b e^{-za} x)|| \leq r(e^{za} b e^{-za})||x||_E \\ &\leq ||b||||x||_E. \end{aligned}$$

It implies that

$$\rho_E(z(ab - ba)x + ...) - \rho_E(bx) \leq ||\rho(bx + z(ab - ba)x + ...)|| \leq ||b||||x||_E.$$

Hence $\rho_E(z(ab - ba)x + ...) \leq ||b||||x||_E + ||\rho(bx)||$.

Thus

$$\rho_E o f'(z) = \rho_E((ab - ba)x + ...) \leq \frac{||b||||x||_E + ||\rho(bx)||}{|z|} \to 0 \ as \ |z| \to \infty.$$

Therefore, $\rho((ab - ba)x) = 0$.

Now suppose that ρ is an $\mathcal{A}$-valued seminorm. By Theorem 2.3 quotient $\mathcal{A}$-module E/N_ρ is a pre-Finsler $\mathcal{A}$-module with the module structure $a(x + N_\rho) = ax + N_\rho$ and $\rho'(x + N_\rho) = \rho(x)$. It is clear that is shown ρ' satisfies in assumption Theorem. Hence, $\rho'((ab - ba)x + N_\rho) = 0$. Now, we have $\rho((ab - ba)x) = \rho'((ab - ba)x + N_\rho) = 0$.

Theorem 3.3 *Let E be a left A-module over a C^*-algebra $\mathcal{A}$ and $\rho : E \to A_+$ be an $\mathcal{A}$-valued Finsler seminorm . Then for every $0 \neq x \in E$ which $\rho(x) \in Z(\mathcal{A})$, I_x is a closed two sided ideal and quotient algebra $\mathcal{A}/I_x$ is commutative, where $Z(\mathcal{A})$ is center of $\mathcal{A}$.*

proof. Let $a \in \mathcal{A}$. Since $a\rho(x) = \rho(x)a$. Thus,

$$\rho(ax)^2 = a\rho(x)^2a^* = |a|^2\rho(x)^2 = \rho(|a|^2\rho(x)x). \tag{1}$$

Define $||a_\rho||_1 = ||\rho(ax)||$ on quotient algebra $\mathcal{A}/I_x$ where $a_\rho = a + I_x$. It is clear that $||.||_1$ is a norm. By using (1) and Lemma 2.1 we conclude

$$\begin{aligned} {||a_\rho||_1}^2 &= ||(|a|^2\rho(x))_\rho||_1 = ||\rho(a^2x)_\rho||_1 \\ &= ||\rho(\rho(a^2x)x)|| \leq ||\rho(a^2x)||||\rho(x)|| \\ &= ||{a_\rho}^2||_1||\rho(x)||. \end{aligned}$$

Hence

$${||a_\rho||_1}^2 \leq ||{a_\rho}^2||_1||\rho(x)||. \tag{2}$$

Let $a \in \mathcal{A}$ and M be maximal commutative subalgebra of $\mathcal{A}_\rho = \mathcal{A}/I_x$ containing a_ρ. Then $(M, ||.||_1)$ is a normed algebra with $||.||_1$ satisfying (2). Let $Sp_K(.)$ and $r_K(.)$ denote respectively the spectrum and the spectral radius in an algebra K. By introduction (1) follows that ${||a_\rho||_1}^{2^n} \leq ||a^{2^n}||_1||\rho(x)||^{2^n-1}$. So that

$$||a_\rho||_1 \leq \limsup {||{a_\rho}^{2^n}||_1}^{1/2^n} ||\rho(x)||^{\frac{2^n-1}{2^n}} = r(a_\rho)||\rho(x)||.$$

[See 3]. By maximality M [4, Theorem 4.6] gives $Sp_M(a_\rho) = Sp_{A_\rho}(a_\rho)$, hence $r_M(a_\rho) = r_{A_\rho}(a_\rho)$. Also $Sp_{A_\rho}(a_\rho) \subset Sp_{A_\rho}(a)$. It follows that

$$||\rho(ax)|| = ||a_\rho||_1 \leq r(a)||\rho(x)||. \tag{$*$}$$

By the lemma 3.2 for every $a, b \in \mathcal{A}$ we have $\rho((ab - ba)x) = 0$. Therefore $ab - ba \in I_x$. Consequently, I_x is a closed two sided ideal of $\mathcal{A}$ [see Theorem 3.1] and quotient algebra $\mathcal{A}/I_x$ is commutative.

Recall that in [6, proposition 1] was proved that every Finsler $\mathcal{A}$-module is a Banach $\mathcal{A}$-module, that is, $||ax||_E \leq ||a||||x||_E$ for all $a \in \mathcal{A}$ and $x \in E$, where $||x||_E = ||\rho(x)||$. The following result gives suitable relation.

Corollary 3.4 *Let E be a left $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$ and ρ an $\mathcal{A}$-valued seminorm on E. Then $||\rho(ax)|| \leq r(a)||\rho(x)||$ for all $x \in E$ which $\rho(x) \in Z(A)$ and all $a \in \mathcal{A}$.*

proof. This follows from the relation (*) in Theorem 3.3.

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