New Algorithm of Solving Nonlinear Singular Third-Order Boundary-Value Problem in the Reproducing Kernel Space

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Abstract

In this paper, we establish iteration methods to compute solution for a class of singular third-order boundary value problems \( \frac{d^3u}{dx^3} = h(u), 0 < x < +\infty, \) and \( u(0) = 0, u(+\infty) = 1, u'(+\infty) = u''(+\infty) = 0, \) where \( h(u) = (1-u)^\lambda g(u), \lambda > 0 \) is a given constant, and \( g(y) \) is a continuous, positive and non-increasing function defined on \((0, 1]\). Representation of the solution is given in the form of series in the reproducing kernel space \( W_2[0, \infty) \). The \( n \)-term approximation \( u_n(x) \) is proved converge to the exact solution. Furthermore, the approximate error of \( u_n(x) \) is monotone decreasing. Some numerical examples have been studied to demonstrate the accuracy of the present method.

Mathematics Subject Classification: 34L16; 65L10

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1 Introduction

The third-order differential equations have attracted considerable attention over the last three decades, and many techniques arose in the studies

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existence of solution of this kind problem, for instance, differential inequality [1,2], topological transversality [3], shooting argument [4], lower and upper solutions method [5], comparable analysis with classical equations [6].

Singular third-order boundary value problem

\[
\begin{aligned}
&\frac{d^3 u}{dx^3} = h(u(x)), \quad 0 < x < +\infty \\
&u(0) = 0, \quad u(+\infty) = 1, \quad u'(+\infty) = u''(+\infty) = 0,
\end{aligned}
\]

(1.1)

where \( h(u(x)) = (1 - u(x))^\lambda g(u(x)) \), \( \lambda > 0 \) is a given constant, and \( g(u(x)) \) is a continuous, positive and non-increasing function defined on \((0,1]\). The singular third-order boundary value problem arises in the study of draining and coating flows. Generally speaking, \( h(u(x)) \) is singular at \( u(x) = 0 \). The simplest and most important one is \( h(u(x)) = 1 - u(x) u'''(x) \). For details, see [4,6,7] and the references therein. In ref [8], they proved that the singular third-order boundary value problems (1.1) has a unique solution. And as we know, up to now, no one gives method to get approximate solution for a singular nonlinear system in literature [8]. The goal of this paper is to use iteration implicit methods for solving singular third-order nonlinear boundary-value problem in ordinary differential equations.

Put \( Lu(x) \equiv \frac{d^3 u(x)}{dx^3} \). Through homogenization of boundary-value condition, then Eq.(1.1) can further be converted into following form:

\[
\begin{aligned}
&Lu(x) = f(u(x)) \\
&u(0) = 0, \quad u(+\infty) = 0, \quad u'(+\infty) = u''(+\infty) = 0
\end{aligned}
\]

(1.2)

where \( x \in (0, \infty) \) and \( u(x) \in W_4[0, \infty) \). Besides \( f(u(x)) \in W_1[0, \infty) \).

2 Several Reproducing Kernel Spaces

1 The reproducing kernel space \( W_2[0, \infty) \)

Inner space \( W_2[0, \infty) \) is defined as \( W_2[0, \infty) = \{ u(x) \mid u, u', u'', u''' \text{ are absolutely continuous real value functions, } u(x), u'(x), u''(x), u'''(x), u^{(4)}(x) \in L^2[0, \infty), u(0) = 0, u(+\infty) = 0, u'(+\infty) = u''(+\infty) = 0 \} \). The inner product in \( W_2[0, \infty) \) is given by

\[
(u(x), v(x))_{W_2} = \int_0^\infty (576uv + 820u'v' + 273u''v'' + 30u'''v''' + u^{(4)}(x)v^{(4)}(x))dx,
\]

(2.1)

\( u(x), v(x) \in W_2[0, \infty) \) and the norm \( ||u||_{W_2} \) is denoted by \( ||u||_{W_2} = \sqrt{(u, u)_{W_2}} \). \( W_2[0, \infty) \) is a reproducing kernel space and corresponding reproducing kernel is

\[ R_x(y) = \]
\[
\begin{aligned}
& \left\{ \begin{array}{l}
c_{21} e^y + c_{22} e^{-y} + c_{23} e^{2y} + c_{24} e^{-2y} + c_{25} e^{3y} + c_{26} e^{-3y} + c_{27} e^4y + c_{28} e^{-4y}, \\
d_{21} e^{-y} + d_{22} e^{-2y} + d_{23} e^{-3y} + d_{24} e^{-4y},
\end{array} \right. \\
& y \leq x, \quad y > x,
\end{aligned}
\]

where
\[
\begin{aligned}
c_{21} &= \frac{e^{-x}}{720}, \\
c_{22} &= \frac{-e^{-4x} (1586-7047 e^x+6032 e^{2x}+18166 e^{3x})}{13685040}, \\
c_{23} &= \frac{e^{-2x}}{720}, \\
c_{24} &= \frac{e^{-x} (13312-50544 e^x+62271 e^{2x}-6032 e^{3x})}{13685040}, \\
c_{25} &= \frac{e^{-3x}}{1680}, \\
c_{26} &= \frac{e^{-4x} (-36288+118774 e^x-117936 e^{2x}+16443 e^{3x})}{31931760}, \\
c_{27} &= \frac{e^{-5x}}{10080}, \\
c_{28} &= \frac{e^{-6x} (76351-217728 e^x+186368 e^{2x}-25984 e^{3x})}{191590560}, \\
d_{21} &= \frac{-e^{-4x} (-1856+7047 e^x-6032 e^{2x}-18166 e^{3x}+19007 e^{5x})}{13685040}, \\
d_{22} &= \frac{-e^{-4x} (-13312+50544 e^x-62271 e^{2x}+6032 e^{3x}+19007 e^{5x})}{13685040}, \\
d_{23} &= \frac{e^{-4x} (-36288+118774 e^x-117936 e^{2x}+16443 e^{3x}+19007 e^{5x})}{31931760}, \\
d_{24} &= \frac{-e^{-4x} (-76351+217728 e^x-186368 e^{2x}+25984 e^{3x}+19007 e^{5x})}{191590560}.
\end{aligned}
\]

Using mathematics 5.0 software package we may prove that \( R_x(y) \in W_2[0, \infty) \) is the reproducing kernel of space \( W_2[0, \infty) \), namely, \( \forall u(x) \in W_2[0, \infty) \), we have \( u(x) = (R_x(y), u(y)) \).

2 The reproducing kernel space \( W_1[0, \infty) \)

The inner space \( W_1[0, \infty) \) is defined by \( W_1[0, \infty) = \{ u(x) \mid u \text{ is absolutely continuous real value function}, \ u' \in L^2[0, \infty) \} \). The inner product and norm in \( W_1[0, \infty) \) are given respectively by
\[
(u(x), v(x))_{W_1} = \int_0^\infty (u(x) + u'(x)) \, dx, \quad \| u \|_{W_1} = \sqrt{(u, u)_{W_1}},
\]
where \( u(x), v(x) \in W_1[0, \infty) \). \( W_1[0, \infty) \) is a reproducing kernel space and its reproducing kernel is
\[
Q_x(y) = \frac{1}{4} (1 + |x - y| e^{-|x-y|}).
\]

3 The solution of \( Eq.(1.2) \)

In this section, the solution of \( Eq.(1.2) \) is given in the reproducing kernel space \( W_2[0, \infty) \).

Note that definitions of \( W_1[0, \infty) \) and \( W_2[0, \infty) \), it is clear that \( L : W_2[0, \infty) \rightarrow W_1[0, \infty) \) is a bounded linear operator. Let \( \varphi_i(x) = Q_{x_i}(x) \), where \( \{x_i\}_{i=1}^\infty \) is dense in the interval \( (0, \infty) \), and \( \psi_i(x) = L^* \varphi_i(x) \), \( L^* \) is the conjugate operator of \( L \), and \( \forall u(x) \in W_1[0, \infty) \), we have
\[
< u(x), \varphi_i(x) > = u(x_i).
\]
\{\overline{\psi_i}(x)\}_{i=1}^{\infty} \text{ derives from Gram-Schmidt orthogonalization process of } \{\psi_i(x)\}_{i=1}^{\infty}, \text{ namely, }
\overline{\psi_i}(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \quad (\beta_{ii} > 0, \ i = 1, 2, \ldots). \tag{3.2}

**Theorem 3.1.** For Eq. (1.2), suppose inverse operator $L^{-1}$ exists, therefore, if $\{x_i\}_{i=1}^{\infty}$ is dense in $[0, \infty)$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2(0, \infty)$.

**Proof.** For each fixed $u(x) \in W_2[0, \infty)$, let $< u(x), \psi_1(x) > = 0, \ (i = 1, 2, \cdots)$, that is, $< u(x), L^* \varphi_i(x) > = (Lu)(x_i) = 0$, note that $\{x_i\}_{i=1}^{\infty}$ is dense in $(0, \infty)$, therefore, $(Lu)(x) = 0$. It follows that $u(x) \equiv 0$ from the existence of $L^{-1}$. \hfill \Box

**Theorem 3.2.** Let $\{x_i\}_{i=1}^{\infty}$ is dense in $(0, \infty)$, because Eq. (1.1) has unique solution (see [10]), then the solution satisfies the form

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(u(x_k)) \overline{\psi_i}(x). \tag{3.3}$$

**Proof.** Assume $u(x)$ be the solution of Eq. (1.2). By theorem 3.1, it is easy to know that $\{\overline{\psi_i}(x)\}_{i=1}^{\infty}$ is the complete normal orthogonal system of $W_2[0, \infty)$, then $u(x)$ is expanded in Fourier series

$$u(x) = \sum_{i=1}^{\infty} < u(x), \overline{\psi_i}(x) > \overline{\psi_i}(x). \tag{3.4}$$

By the form (3.1) and (3.2), we have

$$u(x) = \sum_{i=1}^{\infty} < u(x), \sum_{k=1}^{i} \beta_{ik} \psi_k(x) > \overline{\psi_i}(x)
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} < u(x), \psi_k(x) > \overline{\psi_i}(x)
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} < u(x), L^* \varphi_k(x) > \overline{\psi_i}(x)
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} < Lu(x), \varphi_k(x) > \overline{\psi_i}(x)
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(u(x_k)) \overline{\psi_i}(x). \tag{3.5}$$

\hfill \Box

**Lemma 3.1.** $\psi_i(x) = \frac{d}{dy} R_x(y)\big|_{y=x_i}$ holds.

**Proof.** $\psi_i(x) = (\psi_i(y), R_x(y))_{\omega_2} = (L^* \varphi_i(y), R_x(y))_{\omega_2} = (\varphi_i(y), LR_x(y))_{\omega_2} = LR_x(x_i) = \frac{d}{dy} R_x(y)\big|_{y=x_i}$. \hfill \Box
Let \( \{ \overline{\psi}_i(x) \}_{i=1}^{\infty} \) be the normal orthogonal system derived from Gram-Schmidt orthogonalization process of \( \{ \psi_i(x) \}_{i=1}^{\infty} \). Representation of the solution of Eq.(1.2) can be denoted by

\[
u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x), \tag{3.6}\]

where \( A_i = \sum_{k=1}^{i} \beta_{ik} f(u(x_k)) \). In fact, \( A_i \) is unknow, we will approximate \( A_i \) using know \( A_i \), since \( f(u(x_1)) \) is known on \( x_1 = 0 \), so for a numerical computation, we put \( u_0(x_1) = u(x_1) \), and define the n-term approximation to \( u(x) \) by

\[
u_n(x) = \sum_{i=1}^{n} \overline{A}_i \overline{\psi}_i(x), \tag{3.7}\]

where

\[
\begin{align*}
\overline{A}_1 &= \beta_{11} f(u_0(x_1)) \\
\overline{A}_2 &= \sum_{k=1}^{2} \beta_{2k} f(u_{k-1}(x_k)) \\
&\quad \vdots \\
\overline{A}_n &= \sum_{k=1}^{n} \beta_{nk} f(u_{k-1}(x_k)).
\end{align*}
\tag{3.8}\]

Next, the convergence of \( \nu_n(x) \) will be proved. Now, two Lemmas are given.

**Lemma 3.2.** If \( u(x) \in W_2[0, \infty) \), then there exists \( M > 0 \), such that \( |u(x)| \leq M \| u(x) \|_{W_2[0, \infty)} \).

**Lemma 3.3.** If \( u_n(x) \rightarrow \overline{u}(x) \) \( \text{as} \ n \rightarrow \infty \), \( x_n \rightarrow y(n \rightarrow \infty) \) and \( \| u_n(x) \| \) is bounded, \( f(u(x)) \) is continuous in \( (0, \infty) \), then

\[
f(u_{n-1}(x_n)) \rightarrow f(\overline{u}(y)) \ \text{as} \ n \rightarrow \infty.
\]

**Theorem 3.3.** Suppose that \( \| u_n(x) \| \) is bounded in (3.7), and \( f(u(x)) \in W_1[0, \infty) \), if \( \{ x_i \}_{i=1}^{\infty} \) is dense in \( (0, \infty) \), then the n-term approximate solution \( u_n(x) \) derived from (3.7) converges to the exact solution \( u(x) \) of E.q.(1.2) and

\[
u(x) = \sum_{i=1}^{\infty} \overline{A}_i \overline{\psi}_i(x), \text{ where } \overline{A}_i \text{ is given by (3.8)}.
\]

**Proof.** (1) First, we will prove the convergence of \( u_n(x) \).

By (3.7), we infer that

\[
u_{n+1}(x) = u_n(x) + \overline{A}_{n+1} \overline{\psi}_{n+1}(x). \tag{3.9}\]
From the orthogonality of \( \{ \psi_i(x) \}_{i=1}^\infty \), it follows that
\[
\| u_{n+1} \|_{W_2^4(0,\infty)}^2 = \| u_n \|_{W_2^4(0,\infty)}^2 + (A_{n+1})^2 \\
= \| u_{n-1} \|_{W_2^4(0,\infty)}^2 + (a_n)^2 + (A_{n+1})^2 \\
\ldots \\
= \| u_0 \|_{W_2^4(0,\infty)}^2 + \sum_{i=1}^{n+1} (A_i)^2.
\]
\[\text{(3.10)}\]

From (3.10), we can know sequence \( \| u_n(x) \|_{W_2^4(0,\infty)} \) is monotone decreasing. Due to \( \| u_n(x) \|_{W_2^4(0,\infty)} \) is bounded, \( \| u_n(x) \|_{W_2^4(0,\infty)} \) is convergent as \( n \to \infty \), such that
\[
\sum_{i=1}^{\infty} (A_i)^2 < \infty.
\]
This implies that
\[
\overline{A}_i = \sum_{k=1}^{i} \beta_{ik} f(u_{k-1}(x_k)) \in l^2, \quad (i = 1, 2, \ldots).
\]
\[\text{(3.11)}\]

Without loss of generality, assume \( m > n \), we have
\[
\| u_m(x) - u_n(x) \|_{W_2^4(0,\infty)}^2 = \\
\| u_m(x) - u_{m-1}(x) + u_{m-1}(x) - u_{m-2}(x) + \ldots + u_{n+1}(x) - u_n(x) \|_{W_2^4(0,\infty)}^2 \\
\leq \| u_m(x) - u_{m-1}(x) \|_{W_2^4(0,\infty)}^2 + \ldots + \| u_{n+1}(x) - u_n(x) \|_{W_2^4(0,\infty)}^2 \\
= \sum_{i=n+1}^{m} (A_i)^2 \to 0, \quad (n \to \infty).
\]

Considering the completeness of \( W_4^4(0, \infty) \), \( \exists \overline{u}(x) \in W_2^4(0, \infty) \) such that
\[
u_n(x) \xrightarrow{\| \} \overline{u}(x) \quad \text{as} \quad n \to \infty.
\]

(2) Second, we will prove that \( \overline{u}(x) \) is the solution of \( E.q. (1.2) \).
By lemma 3.3 and (1) of theorem (3.3), we may know \( u_n(x) \) is convergent uniformly to \( \overline{u}(x)(n \to \infty) \). It follows that, on taking limits for \( n \) in (3.7), we have
\[
\overline{u}(x) = \sum_{i=1}^{\infty} \overline{A}_i \overline{\psi}_i(x).
\]

Since
\[
(L\overline{u})(x) = \sum_{i=1}^{\infty} \overline{A}_i (L\overline{\psi}_i, \varphi_j) = \sum_{i=1}^{\infty} \overline{A}_i (\overline{\psi}_i, L^* \varphi_j) = \sum_{i=1}^{\infty} \overline{A}_i (\overline{\psi}_i, \psi_j),
\]
it follows that
\[
\sum_{j=1}^{n} \beta_{nj}(Lu)(x_j) = \sum_{i=1}^{\infty} A_i(\bar{\psi}_i, \sum_{j=1}^{n} \beta_{nj}\psi_j) = \sum_{i=1}^{\infty} A_i(\bar{\psi}_i, \bar{\psi}_n) = A_n.
\]

If \( n = 1 \), then
\[
(Lu)(x_1) = f(u_0(x_1)). \tag{3.12}
\]

If \( n = 2 \), then
\[
\beta_{21}(Lu)(x_1) + \beta_{22}(Lu)(x_2) = \beta_{21}f(u_0(x_1)) + \beta_{22}f(u_1(x_2)). \tag{3.13}
\]

From (3.12) and (3.13), it is clear that
\[
(Lu)(x_2) = f(u_1(x_2)).
\]

Furthermore, it is easy to see by induction that
\[
(Lu)(x_j) = f(u_{j-1}(x_j)), \quad j = 1, 2, \cdots. \tag{3.14}
\]

Since \( \{x_i\}_{i=1}^{\infty} \) is dense in \((0, \infty)\), for \( \forall y \in (0, \infty) \), there exists subsequence \( \{x_{n_j}\} \), such that
\[
x_{n_j} \rightarrow y \quad as \quad j \rightarrow \infty.
\]

Hence, let \( j \rightarrow \infty \) in (3.14), by the convergence of \( u_n(x) \) and Lemma 3.3, we have
\[
(Lu)(y) = f(\bar{u}(y)). \tag{3.15}
\]

That is, \( \bar{u}(x) \) is the solutions of Eq. (1.2) and
\[
u(x) = \sum_{i=1}^{\infty} A_i\bar{\psi}_i(x) \tag{3.16}
\]

\[\square\]

**Theorem 3.4.** Assume \( u(x) \) is the solution of Eq. (1.2), \( r_n \) is the approximate error of \( u_n(x) \), and \( r_n = \|u(x) - u_n(x)\|_{W_2} \), where \( u_n(x) \) is given by (3.7). Then the error \( r_n \) is monotone decreasing in the sense of \( \| \cdot \|_{W_2[0, \infty)} \).

**Proof.** From (3.7), (3.16), it follows that
\[
\| r_n(x) \|_{W_2}^2 = \| \sum_{i=n+1}^{\infty} A_i\bar{\psi}_i(x) \|_{W_2}^2 = \sum_{i=n+1}^{\infty} (A_i)^2. \tag{3.17}
\]

(3.17) shows that the error \( r_n \) is monotone decreasing in the sense of \( \| \cdot \|_{W_2[0, \infty)} \). \[\square\]
4 Numerical example

Example

Considering equation

\[
\begin{cases}
    u''(x) = \frac{1-u(x)}{u(x)}, & 0 < x < \infty \\
u(0) = 0, u(+\infty) = 1, u'(+) = u''(+) = 0,
\end{cases}
\]

where \( x \in (0, \infty) \). The true solution is \( u(x) = x ((x + e^{-x})^{-1}) \). Using our method, through homogenization of boundary-value condition, let \( \overline{u}(x) = 1 - u(x) - e^{-x} \), then \( \overline{u}(0) = 0, \overline{u}(+\infty) = 0, \overline{u}'(+) = \overline{u}''(+) = 0 \), we choose 40 points on \((0, \infty)\) and obtain approximate solution \( \overline{u}_{40}(x) \). The numerical results are presented in Table 1. From the table, it illustrates that the method given in the paper is efficient.

<table>
<thead>
<tr>
<th>Node</th>
<th>True solution ( \overline{u}(x) )</th>
<th>Approximate solution ( \overline{u}_{40} )</th>
<th>Absolute error</th>
<th>Relative error</th>
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References


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