Fixed Points of Demi-Closed Mappings

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Abstract. Let \( f \) and \( g \) be two non expansive mappings on a Reflexive Banach space satisfying certain inequality. Here first we will prove that \( \frac{f+g}{2} \) has a fixed point if \( I - \frac{f+g}{2} \) is demi-closed. Then we can extend the result to a common fixed point theorem for \( f \) and \( g \) using the assumption that \( f - g \) is demi-closed. Also we can prove that \( f \) and \( g \) has a common fixed point if \( I - f \) and \( I - g \) are demi-closed.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Nonexpansive mappings, Reflexive Banach space, demi-closed mappings, fixed points

1. INTRODUCTION

Throughout this paper we shall suppose that \((X,\|\cdot\|)\) is a real Banach space, \(R\) denote the set of real numbers and \(N\) denote the set of natural numbers. Let \(C \subseteq X\) be a nonempty unbounded closed convex set. We say that \(h : C \to X\) is demi-closed if for any sequence \(\{x_n\} \subseteq C\) weakly convergent to an element \(x_* \in C\) with \(\{h(x_n)\}\) norm-convergent to an element \(y_*\), then \(h(x_*) = y_*\) [1]. Also recall that \(h : C \to X\) is nonexpansive if \(\|h(x) - h(y)\| \leq \|x - y\|\) \(\forall x, y \in C\). The mapping \(h : C \to X\) is said to be Lipschitzian if there exist a constant \(\rho > 0\) such that \(\|h(x) - h(y)\| \leq \rho \|x - y\|\) \(\forall x, y \in C\). If \(\rho < 1\) the mapping \(h\) is called contractive and by the well known result called Banach’s contraction mapping principle if \(h : X \to X\) is a contraction \(h\) has a unique fixed point in \(X\) [4].
A semi-inner-product on $X$ is a function $[\cdot,\cdot]: X \times X \to \mathbb{R}$ satisfying the following properties,

(i) $[x + y, z] = [x, z] + [y, z]$
(ii) $[\lambda x, y] = \lambda [x, y]$
(iii) $[x, x] > 0$ for $x \neq 0$
(iv) $[[x, y]]^2 \leq [x, x][y, y]$ \ \forall x, y, z \in X, \lambda \in \mathbb{R}$

A semi-inner-product space is a normed linear space with the norm $\|x\| = [x, x]^{1/2}$ [2]. Also it is possible to define a semi-inner-product such that $[x, x] = \|x\|^2$, where $\|\cdot\|$ is the norm given in $X$. By the proof of theorem(1) in [2] this semi-inner-product can be defined so that it satisfy, $[x, \lambda y] = \lambda [x, y] \ \forall x, y, \in X, \lambda \in \mathbb{R}$. Now we will consider a similar function $G: X \times X \to \mathbb{R}$ and then by using the concept of demi-closed mappings we will prove a common fixed point theorem.

2. MAIN RESULTS

Remark(1): Let $(X, \|\cdot\|)$ be a real Banach space and $G: X \times X \to \mathbb{R}$ be a mapping such that,

$\forall x, y, z \in X, \lambda \in \mathbb{R}$ \ 
(i) $G(x + y, z) = G(x, z) + G(y, z) \\
(ii) G(\lambda x, y) = \lambda G(x, y) \\
(iii) \|x\|^2 \leq G(x, x) \\
(iv) \exists M > 0$ such that $|G(x, y)| \leq M \|x\| \|y\|$

Then in theorem(3.2) of [3], G.Isac and S.Z.Nemeth proved a fixed point theorem for a nonexpansive mapping $f$ satisfying certain inequality if $X$ is Reflexive. Now first we will extend the result to the mapping $\frac{f + g}{2}$ where $f$ and $g$ are two nonexpansive mappings.

Theorem(1): Let $(X, \|\cdot\|)$ be a Reflexive Banach space and $C \subseteq X$ a nonempty unbounded closed convex set. Let $f : C \to C$ and $g : C \to C$ be two nonexpansive mappings such that $I - \frac{f + g}{2}$ is demi-closed. If

\[ \lim \sup_{\|x\| \to \infty} \frac{G(f(x) - x_0, x)}{\|x\|^2} + \lim \sup_{\|x\| \to \infty} \frac{G(g(x) - x_0, x)}{\|x\|^2} < 1 \text{ for some } x_0 \in C, (*) \]

then $\frac{f + g}{2}$ has a fixed point in $C$. 

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**Proof:** We know that \( f \) and \( g \) are bounded mappings (That is, \( f(D) \) and \( g(D) \) are bounded whenever \( D \subseteq C \) is bounded).

Let \( \{\lambda_n\} \) be a sequence in (0,1) such that \( \lim_{n \to \infty} \lambda_n = 0 \)

For all \( n \in \mathbb{N} \), define, \( \phi_n : C \to C \) by

\[
\phi_n(x) = \frac{(1-\lambda_n)}{2} f(x) + \frac{(1-\lambda_n)}{2} g(x) + \lambda_n x_0.
\]

Then since \( 1 - \lambda_n < 1 \), \( \phi_n \) is contractive and hence \( \phi_n \) has a unique fixed point \( x_n \in C \). That is \( \forall n \in \mathbb{N} \), there exist \( x_n \in C \) such that \( \phi_n(x_n) = x_n \). Now we will prove that \( \{x_n\}_{n \in \mathbb{N}} \) is a bounded sequence.

If possible suppose that \( \{x_n\} \) is not bounded. From (*) we have lim sup \( \|x\| \to \infty \frac{G(f(x) - x_0, x)}{\|x\|^2} < 1 \) and lim sup \( \|x\| \to \infty \frac{G(g(x) - x_0, x)}{\|x\|^2} < 1 \).

Therefore there exist \( \beta_1 \in (0,1) \) and \( k_1 > 0 \) such that, \( G(f(x) - x_0, x) \leq \beta_1 \|x\|^2 \forall x \in C \) with \( \|x\| > k_1 \). Also there exist \( \beta_2 \in (0,1) \) and \( k_2 > 0 \) such that, \( G(g(x) - x_0, x) \leq \beta_2 \|x\|^2 \forall x \in C \) with \( \|x\| > k_2 \).

Let \( \beta = \max \{\beta_1, \beta_2\} \) and \( k = \max \{k_1, k_2\} \). Then, \( G(f(x) - x_0, x) \leq \beta \|x\|^2 \) and \( G(g(x) - x_0, x) \leq \beta \|x\|^2 \forall x \in C \) such that \( \|x\| > k \).

For \( n \in \mathbb{N} \) large enough we have,

\[
\|x_n\|^2 \leq G(x_n, x_0) = G(\phi_n(x_n), x_0)
\]

\[
= G\left(\frac{(1-\lambda_n)}{2} f(x_n) + \frac{(1-\lambda_n)}{2} g(x_n) + \lambda_n x_0, x_n\right)
\]

\[
= G\left(\frac{(1-\lambda_n)}{2} f(x_n) + \frac{(1-\lambda_n)}{2} g(x_n) - (1-\lambda_n)x_0 + x_0, x_n\right)
\]

\[
= G\left(\frac{(1-\lambda_n)}{2} (f(x_n) - x_0) + \frac{(1-\lambda_n)}{2} (g(x_n) - x_0) + x_0, x_n\right)
\]

\[
= \frac{(1-\lambda_n)}{2} G(f(x_n) - x_0, x_n) + \frac{(1-\lambda_n)}{2} G(g(x_n) - x_0, x_n) + G(x_0, x_n)
\]

\[
\leq \frac{(1-\lambda_n)}{2} \beta \|x_n\|^2 + \frac{(1-\lambda_n)}{2} \beta \|x_n\|^2 + M \|x_0\| \|x_n\|
\]

Dividing by \( \|x_n\|^2 \), we have

\[
1 \leq \frac{(1-\lambda_n)}{2} \beta + \frac{(1-\lambda_n)}{2} \beta + M \frac{\|x_0\|}{\|x_n\|}
\]
Letting \( n \to \infty \) we have \( 1 \leq \beta \) (since \( \{x_n\} \) is unbounded) which is a contradiction. Thus \( \{x_n\} \) is bounded. Since \( X \) is reflexive we may assume that \( \{x_n\} \) is weakly convergent to an element \( x_\ast \in C \). (For, consider a weakly convergent subsequence of \( \{x_n\} \).)

Now consider,
\[
\phi_n(x_n) = x_n = \frac{1-\lambda_n}{2} f(x_n) + \frac{1-\lambda_n}{2} g(x_n) + \lambda_n x_0
\]
\[
\Rightarrow \left\| x_n - \frac{f(x_n)}{2} + \frac{x_n - g(x_n)}{2} \right\| \leq \frac{\lambda_n}{2} \left\| f(x_n) \right\| + \frac{\lambda_n}{2} \left\| g(x_n) \right\| + \lambda_n \left\| x_0 \right\|
\]

Letting \( n \to \infty \),
\[
\left\| x_n - \frac{f(x_n)}{2} + \frac{x_n - g(x_n)}{2} \right\| \to 0
\]
(Since \( \{x_n\}, \{f(x_n)\} \) and \( \{g(x_n)\} \) are bounded)

Then since \( I - \frac{(f+g)}{2} \) is demi-closed,
\[
\left( I - \frac{(f+g)}{2} \right)(x_\ast) = 0 \Rightarrow \frac{(f+g)}{2}(x_\ast) = x_\ast
\]

Hence \( x_\ast \) is a fixed point of \( \frac{(f+g)}{2} \).

**Corollary (1.1):** Taking \( f = g \) in the proof of theorem (1) we will get theorem (3.2) of [3].

**Remark (2):** If we restrict the existence of the constant \( M \) in the definition of \( G \) as \( M > 1 \) we can prove the following two corollaries.

**Corollary (1.2):** Let \( (X, \| \cdot \|) \) be a Reflexive Banach space and \( C \subseteq X \) a nonempty unbounded closed convex set. Let \( f : C \to C \) and \( g : C \to C \) be two nonexpansive mappings such that \( I - \frac{(f+g)}{2} \) and \( f-g \) are demi-closed. If
\[
\limsup \frac{G(f(x) - x_0, x)}{\| x \|^2} + \limsup \frac{G(g(x) - x_0, x)}{\| x \|^2} < 1 \text{ for some } x_0 \in C,
\]
then \( f \) and \( g \) has a common fixed point in \( C \).

**Proof:** From theorem (1) we have a bounded sequence \( \{x_n\} \) in \( C \) such that \( \{x_n\} \) converges weakly to an element \( x_\ast \in C \) and \( x_\ast \) is a fixed point of \( \frac{(f+g)}{2} \).

That is we have \( \frac{(f+g)}{2}(x_\ast) = x_\ast \) (1)

Now we will prove that
\[
\lim_{n \to \infty} \left\| f(x_n) - g(x_n) \right\| = 0
\]
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If possible let \( \lim_{n \to \infty} \|f(x_n) - g(x_n)\| = K > 0 \). Then consider,

\[
\|f(x_n) - g(x_n)\|^2 \leq G(f(x_n) - g(x_n), f(x_n) - g(x_n)) \\
\leq G(f(x_n), f(x_n) - g(x_n)) + G(g(x_n), f(x_n) - g(x_n)) \\
\leq M \|f(x_n) - g(x_n)\| + M \|g(x_n)\| \|f(x_n) - g(x_n)\|
\]

Taking limit as \( n \to \infty \),

\[
K^2 \leq MK \lim_{n \to \infty} \|f(x_n)\| + MK \lim_{n \to \infty} \|g(x_n)\|
\]

\( \Rightarrow K \leq M \lim_{n \to \infty} \|f(x_n)\| + M \lim_{n \to \infty} \|g(x_n)\| \), since \( K > 0 \)

\( \Rightarrow \lim_{n \to \infty} \|f(x_n)\| - \lim_{n \to \infty} \|g(x_n)\| \leq K \leq M \lim_{n \to \infty} \|f(x_n)\| + M \lim_{n \to \infty} \|g(x_n)\| \)

\( \Rightarrow (1 - M) \lim_{n \to \infty} \|f(x_n)\| \leq (1 + M) \lim_{n \to \infty} \|g(x_n)\| \)

\( \Rightarrow \lim_{n \to \infty} \|f(x_n)\| = \lim_{n \to \infty} \|g(x_n)\| \), since \( M > 1 \)

But then \( \lim_{n \to \infty} \|f(x_n) - g(x_n)\| = 0 \Rightarrow K = 0 \), which is a contradiction.

Therefore \( \lim_{n \to \infty} \|f(x_n) - g(x_n)\| = 0 \).

Then since \( f - g \) is demi-closed we have,

\( (f - g)(x_n) = 0 \)

That is, \( f(x_n) = g(x_n) \). But then from (1) \( f(x_n) = g(x_n) = x_n \).

Hence \( x_n \) is a common fixed point of \( f \) and \( g \).

**Corollary (1.3):** Let \( \left( X, \| \cdot \| \right) \) be a Reflexive Banach space and \( C \subseteq X \) a nonempty unbounded closed convex set. Let \( f : C \to C \) and \( g : C \to C \) be two nonexpansive mappings such that \( I - f \) and \( I - g \) are demi-closed. If

\[
\limsup_{x \in C} \frac{G(f(x) - x_0, x)}{\|x\|^2} + \limsup_{x \in C} \frac{G(g(x) - x_0, x)}{\|x\|^2} < 1 \text{ for some } x_0 \in C,
\]

then \( f \) and \( g \) has a common fixed point in \( C \).

**Proof:** Similarly proceeding in theorem (1) we can construct a bounded sequence \( \{x_n\} \) in \( C \) such that \( \lim_{n \to \infty} \left\| \frac{f(x_n)}{2} + \frac{g(x_n)}{2} - \frac{x_n}{2} \right\| = 0 \).
\[ \lim_{n \to \infty} \|x_n - f(x_n) + x_n - g(x_n)\| = 0 \]

\[ \lim_{n \to \infty} \|x_n - f(x_n)\| - \lim_{n \to \infty} \|g(x_n) - x_n\| \leq \lim_{n \to \infty} \|x_n - f(x_n) + x_n - g(x_n)\| = 0 \]

\[ \lim_{n \to \infty} \|x_n - f(x_n)\| = \lim_{n \to \infty} \|g(x_n) - x_n\| \] (2)

If possible suppose that \( \lim_{n \to \infty} \|x_n - f(x_n)\| = K > 0 \).

Then consider,

\[ \|x_n - f(x_n)\|^2 \leq G(x_n, x_n - f(x_n)) \]

\[ \leq G(x_n, x_n - f(x_n)) + G(-f(x_n), x_n - f(x_n)) \]

\[ \leq M\|x_n\|\|x_n - f(x_n)\| + M\|f(x_n)\|\|x_n - f(x_n)\| \]

Taking limit as \( n \to \infty \),

\[ K^2 \leq MK \lim_{n \to \infty} \|x_n\|^2 + MK \lim_{n \to \infty} \|f(x_n)\| \]

\[ \Rightarrow K \leq M \lim_{n \to \infty} \|x_n\| + M \lim_{n \to \infty} \|f(x_n)\|, \text{since } K > 0 \]

\[ \Rightarrow \lim_{n \to \infty} \|x_n\| - \lim_{n \to \infty} \|f(x_n)\| \leq K \leq M \lim_{n \to \infty} \|x_n\| + M \lim_{n \to \infty} \|f(x_n)\| \]

\[ \Rightarrow (1 - M) \lim_{n \to \infty} \|x_n\| \leq (1 + M) \lim_{n \to \infty} \|f(x_n)\| \]

\[ \Rightarrow \lim_{n \to \infty} \|x_n\| \leq \frac{(1 + M)}{(1 - M)} \lim_{n \to \infty} \|f(x_n)\| \]

\[ \Rightarrow \lim_{n \to \infty} \|x_n\| = 0 = \lim_{n \to \infty} \|f(x_n)\|, \text{since } M > 1 \]

But then \( \lim_{n \to \infty} \|x_n - f(x_n)\| = 0 \Rightarrow K = 0 \) which is a contradiction.

Therefore \( \lim_{n \to \infty} \|x_n - f(x_n)\| = 0 \). Then from (2) \( \lim_{n \to \infty} \|x_n - g(x_n)\| = 0 \)

Since \( I - f \) and \( I - g \) are demi-closed we have \( (I - f)(x_*) = 0 \) and \( (I - g)(x_*) = 0 \)

\[ \Rightarrow f(x_*) = x_* = g(x_*) \]

Hence \( x_* \) is a common fixed point of \( f \) and \( g \).

REFERENCES


Received: November 13, 2007