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Fixed Points of Demi-Closed Mappings

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Abstract. Let f and g be two non expansive mappings on a Reflexive Banach space satisfying certain inequality. Here first we will prove that $\frac{(f+g)}{2}$ has a fixed point if $I-\frac{(f+g)}{2}$ is demi-closed. Then we can extend the result to a common fixed point theorem for f and g using the assumption that $f-g$ is demi-closed. Also we can prove that f and g has a common fixed point if $I-f$ and $I-g$ are demi-closed.

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1. INTRODUCTION

Throughout this paper we shall suppose that $(X,\|.\|)$ is a real Banach space, R denote the set of real numbers and N denote the set of natural numbers. Let $C \subseteq X$ be a nonempty unbounded closed convex set. We say that $h: C \to X$ is demi-closed if for any sequence $\{x_n\} \subseteq C$ weakly convergent to an element $x_* \in C$ with $\{h(x_n)\}$ norm-convergent to an element y_* ,then $h(x_*) = y_*$ [1]. Also recall that $h: C \to X$ is nonexpansive if $\|h(x)-h(y)\| \le \|x-y\| \quad \forall x, y \in C$. The mapping $h: C \to X$ is said to be Lipschitzian if there exist a constant $\rho > 0$ such that $\|h(x)-h(y)\| \le \rho\|x-y\| \quad \forall x, y \in C$. If $\rho < 1$ the mapping h is called contractive and by the well known result called Banach's contraction mapping principle if $h: X \to X$ is a contraction h has a unique fixed point in X [4].

A semi-inner-product on X is a function $[\,,\,]: X \times X \to R$ satisfying the following properties,

(i) $[x+y, z] = [x, z] + [y, z]$

(ii) $[\lambda x, y] = \lambda [x, y]$

(iii) $[x, x] > 0$ for $x \neq 0$

(iv) $|[x, y]|^2 \leq [x, x][y, y] \quad \forall x, y, z \in X, \lambda \in R$

A semi-inner-product space is a normed linear space with the norm $\|x\|_s = [x, x]^{1/2}$ [2]. Also it is possible to define a semi-inner-product such that $[x, x] = \|x\|^2$,where $\|\,.\,\|$ is the norm given in X .By the proof of theorem(1) in [2] this semi-inner-product can be defined so that it satisfy, $[x, \lambda y] = \lambda [x, y] \quad \forall x, y \in X, \lambda \in R$. Now we will consider a similar function $G : X \times X \to R$ and then by using the concept of demi-closed mappings we will prove a common fixed point theorem.

2. MAIN RESULTS

Remark(1): Let $(X, \|.\|)$ be a real Banach space and $G : X \times X \to R$ be a mapping such that,

$\forall x, y, z \in X, \lambda \in R \quad (i) \quad G(x+y, z) = G(x, z) + G(y, z)$

(ii) $G(\lambda x, y) = \lambda G(x, y)$

(iii) $\|x\|^2 \leq G(x, x)$

(iv) $\exists\, M > 0$ such that $|G(x, y)| \leq M \|x\| \, \|y\|$

Then in theorem(3.2) of [3], G.Isac and S.Z.Nemeth proved a fixed point theorem for a nonexpansive mapping f satisfying certain inequality if X is Reflexive**.** Now first we will extend the result to the mapping $\frac{(f+g)}{2}$ where f and g are two non expansive mappings.

Theorem(1): Let $(X, \|.\|)$ be a Reflexive Banach space and $C \subseteq X$ a nonempty unbounded closed convex set. Let $f : C \to C$ and $g : C \to C$ be two nonexpansive mappings such that $I - \frac{(f+g)}{2}$ is demi-closed. If

$$\limsup_{\substack{\|x\| \to \infty \\ x \in C}} \frac{G(f(x) - x_0, x)}{\|x\|^2} + \limsup_{\substack{\|x\| \to \infty \\ x \in C}} \frac{G(g(x) - x_0, x)}{\|x\|^2} < 1 \text{ for some } x_0 \in C, \quad (*)$$

then $\frac{(f+g)}{2}$ has a fixed point in C .

Proof: We know that f and g are bounded mappings (That is, $f(D)$ and $g(D)$ are bounded whenever $D \subseteq C$ is bounded).

Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\lim_{n \to \infty} \lambda_n = 0$

For all $n \in \mathrm{N}$, define, $\phi_n : C \to C$ by

$$\phi_n(x) = \frac{(1-\lambda_n)}{2} f(x) + \frac{(1-\lambda_n)}{2} g(x) + \lambda_n x_0 .$$

Then since $1-\lambda_n < 1$, ϕ_n is contractive and hence ϕ_n has a unique fixed point $x_n \in C$. That is $\forall n \in \mathrm{N}$, there exist $x_n \in C$ such that $\phi_n(x_n) = x_n$ .Now we will prove that $\{x_n\}_{n\in \mathrm{N}}$ is a bounded sequence.

If possible suppose that $\{x_n\}$ is not bounded. From (*) we have $\limsup_{\substack{\|x\| \to \infty \\ x \in C}} \frac{G(f(x)-x_0, x)}{\|x\|^2} < 1$ and $\limsup_{\substack{\|x\| \to \infty \\ x \in C}} \frac{G(g(x)-x_0, x)}{\|x\|^2} < 1$.

Therefore there exist $\beta_1 \in (0,1)$ and $k_1 > 0$ such that, $G(f(x)-x_0, x) \le \beta_1 \|x\|^2$ $\forall x \in C$ with $\|x\| > k_1$. Also there exist $\beta_2 \in (0,1)$ and $k_2 > 0$ such that , $G(g(x)-x_0, x) \le \beta_2 \|x\|^2$ $\forall x \in C$ with $\|x\| > k_2$.

Let $\beta = \max\{\beta_1, \beta_2\}$ and $k = \max\{k_1, k_2\}$. Then, $G(f(x)-x_0, x) \le \beta \|x\|^2$ and $G(g(x)-x_0, x) \le \beta \|x\|^2$ $\forall x \in C$ such that $\|x\| > k$.

For $n \in \mathrm{N}$ large enough we have,

$$\begin{aligned}
\|x_n\|^2 &\le G(x_n, x_n) = G(\phi_n(x_n), x_n) \\
&= G\left(\frac{(1-\lambda_n)}{2} f(x_n) + \frac{(1-\lambda_n)}{2} g(x_n) + \lambda_n x_0, x_n\right) \\
&= G\left(\frac{(1-\lambda_n)}{2} f(x_n) + \frac{(1-\lambda_n)}{2} g(x_n) - (1-\lambda_n) x_0 + x_0, x_n\right) \\
&= G\left(\frac{(1-\lambda_n)}{2} (f(x_n) - x_0) + \frac{(1-\lambda_n)}{2} (g(x_n) - x_0) + x_0, x_n\right) \\
&= \frac{(1-\lambda_n)}{2} G(f(x_n) - x_0, x_n) + \frac{(1-\lambda_n)}{2} G(g(x_n) - x_0, x_n) + G(x_0, x_n) \\
&\le \frac{(1-\lambda_n)}{2} \beta \|x_n\|^2 + \frac{(1-\lambda_n)}{2} \beta \|x_n\|^2 + M \|x_0\| \|x_n\|
\end{aligned}$$

Dividing by $\|x_n\|^2$, we have

$$1 \le \frac{(1-\lambda_n)}{2} \beta + \frac{(1-\lambda_n)}{2} \beta + M \frac{\|x_0\|}{\|x_n\|}$$

Letting $n \to \infty$ we have $1 \le \beta$ (since $\{x_n\}$ is unbounded) which is a contradiction. Thus $\{x_n\}$ is bounded. Since X is reflexive we may assume that $\{x_n\}$ is weakly convergent to an element $x_* \in C$ (For, consider a weakly convergent subsequence of $\{x_n\}$).

Now consider,

$$\phi_n(x_n) = x_n = \frac{(1-\lambda_n)}{2} f(x_n) + \frac{(1-\lambda_n)}{2} g(x_n) + \lambda_n x_0$$

$$\Rightarrow \left\| \frac{x_n}{2} - \frac{f(x_n)}{2} + \frac{x_n}{2} - \frac{g(x_n)}{2} \right\| \le \frac{\lambda_n}{2} \| f(x_n) \| + \frac{\lambda_n}{2} \| g(x_n) \| + \lambda_n \| x_n \|$$

Letting $n \to \infty$, $\left\| \frac{x_n}{2} - \frac{f(x_n)}{2} + \frac{x_n}{2} - \frac{g(x_n)}{2} \right\| \to 0$ (Since $\{x_n\}, \{f(x_n)\}$ and $\{g(x_n)\}$ are bounded)

Then since $I - \frac{(f+g)}{2}$ is demi-closed $\left(I - \frac{(f+g)}{2} \right)(x_*) = 0 \Rightarrow \frac{(f+g)}{2}(x_*) = x_*$

Hence x_* is a fixed point of $\frac{(f+g)}{2}$.

Corollary(1.1): Taking $f = g$ in the proof of theorem(1) we will get theorem(3.2) of [3].

Remark(2): If we restrict the existence of the constant M in the definition of G as $M > 1$ we can prove the following two corollaries.

Corollary(1.2): Let $(X, \|.\|)$ be a Reflexive Banach space and $C \subseteq X$ a nonempty unbounded closed convex set. Let $f : C \to C$ and $g : C \to C$ be two nonexpansive mappings such that $I - \frac{(f+g)}{2}$ and $f - g$ are demi-closed. If

$$\limsup_{\substack{\|x\| \to \infty \\ x \in C}} \frac{G(f(x) - x_0, x)}{\|x\|^2} + \limsup_{\substack{\|x\| \to \infty \\ x \in C}} \frac{G(g(x) - x_0, x)}{\|x\|^2} < 1 \text{ for some } x_0 \in C,$$

then f and g has a common fixed point in C .

Proof: From theorem(1) we have a bounded sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to an element $x_* \in C$ and x_* is a fixed point of $\frac{(f+g)}{2}$.

That is we have $\frac{(f+g)}{2}(x_*) = x_*$ (1)

Now we will prove that $\lim_{n \to \infty} \| f(x_n) - g(x_n) \| = 0$.

If possible let $\lim_{n\to\infty} \|f(x_n)-g(x_n)\| = K > 0$ .Then consider,

$$\|f(x_n)-g(x_n)\|^2 \le G(f(x_n)-g(x_n), f(x_n)-g(x_n))$$
$$\le G(f(x_n), f(x_n)-g(x_n)) + G(-g(x_n), f(x_n)-g(x_n))$$
$$\le M\|f(x_n)\|\|f(x_n)-g(x_n)\| + M\|g(x_n)\|\|f(x_n)-g(x_n)\|$$

Taking limit as $n\to\infty$,

$$K^2 \le MK \lim_{n\to\infty} \|f(x_n)\| + MK \lim_{n\to\infty} \|g(x_n)\|$$

$$\Rightarrow K \le M \lim_{n\to\infty} \|f(x_n)\| + M \lim_{n\to\infty} \|g(x_n)\|, \text{since } K>0$$

$$\Rightarrow \lim_{n\to\infty} \|f(x_n)\| - \lim_{n\to\infty} \|g(x_n)\| \le K \le M \lim_{n\to\infty} \|f(x_n)\| + M \lim_{n\to\infty} \|g(x_n)\|$$

$$\Rightarrow (1-M) \lim_{n\to\infty} \|f(x_n)\| \le (1+M) \lim_{n\to\infty} \|g(x_n)\|$$

$$\Rightarrow \quad \lim_{n\to\infty} \|f(x_n)\| \le \frac{(1+M)}{(1-M)} \lim_{n\to\infty} \|g(x_n)\|$$

$$\Rightarrow \lim_{n\to\infty} \|f(x_n)\| = 0 = \lim_{n\to\infty} \|g(x_n)\|, \text{since } M > 1$$

But then $\lim_{n\to\infty} \|f(x_n)-g(x_n)\| = 0 \Rightarrow K = 0$, which is a contradiction.

Therefore $\lim_{n\to\infty} \|f(x_n)-g(x_n)\| = 0$.

Then since $f-g$ is demi-closed we have,

$$(f-g)(x_*) = 0$$

That is, $f(x_*) = g(x_*)$ .But then from (1) $f(x_*) = g(x_*) = x_*$.

Hence x_* is a common fixed point of f and g.

Corollary(1.3): Let $(X, \|.\|)$ be a Reflexive Banach space and $C \subseteq X$ a nonempty unbounded closed convex set. Let $f: C\to C$ and $g: C\to C$ be two nonexpansive mappings such that $I-f$ and $I-g$ are demi-closed. If

$$\limsup_{\substack{\|x\|\to\infty \\ x\in C}} \frac{G(f(x)-x_0, x)}{\|x\|^2} + \limsup_{\substack{\|x\|\to\infty \\ x\in C}} \frac{G(g(x)-x_0, x)}{\|x\|^2} < 1 \text{ for some } x_0 \in C,$$

then f and g has a common fixed point in C.

Proof: Similarly proceeding in theorem (1) we can construct a bounded sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \left\| \frac{x_n}{2} - \frac{f(x_n)}{2} + \frac{x_n}{2} - \frac{g(x_n)}{2} \right\| = 0$

$$\Rightarrow \lim_{n\to\infty} \|x_n - f(x_n) + x_n - g(x_n)\| = 0$$

$$\Rightarrow \lim_{n\to\infty} \|x_n - f(x_n)\| - \lim_{n\to\infty} \|g(x_n) - x_n\| \le \lim_{n\to\infty} \|x_n - f(x_n) + x_n - g(x_n)\| = 0$$

$$\Rightarrow \lim_{n\to\infty} \|x_n - f(x_n)\| = \lim_{n\to\infty} \|g(x_n) - x_n\| \qquad (2)$$

If possible suppose that $\lim_{n\to\infty} \|x_n - f(x_n)\| = K > 0$.

Then consider,

$$\begin{aligned}\|x_n - f(x_n)\|^2 &\le G(x_n - f(x_n), x_n - f(x_n))\\ &\le G(x_n, x_n - f(x_n)) + G(-f(x_n), x_n - f(x_n))\\ &\le M\|x_n\|\|x_n - f(x_n)\| + M\|f(x_n)\|\,\|x_n - f(x_n)\|\end{aligned}$$

Taking limit as $n \to \infty$,

$$K^2 \le MK \lim_{n\to\infty} \|x_n\| + MK \lim_{n\to\infty} \|f(x_n)\|$$

$$\Rightarrow K \le M \lim_{n\to\infty} \|x_n\| + M \lim_{n\to\infty} \|f(x_n)\|, \text{since } K > 0$$

$$\Rightarrow \lim_{n\to\infty} \|x_n\| - \lim_{n\to\infty} \|f(x_n)\| \le K \le M \lim_{n\to\infty} \|x_n\| + M \lim_{n\to\infty} \|f(x_n)\|$$

$$\Rightarrow (1-M) \lim_{n\to\infty} \|x_n\| \le (1+M) \lim_{n\to\infty} \|f(x_n)\|$$

$$\Rightarrow \lim_{n\to\infty} \|x_n\| \le \frac{(1+M)}{(1-M)} \lim_{n\to\infty} \|f(x_n)\|$$

$$\Rightarrow \lim_{n\to\infty} \|x_n\| = 0 = \lim_{n\to\infty} \|f(x_n)\|, \text{since } M > 1$$

But then $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0 \Rightarrow K = 0$ which is a contradiction.

Therefore $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0$. Then from (2) $\lim_{n\to\infty} \|x_n - g(x_n)\| = 0$

Since $I - f$ and $I - g$ are demi-closed we have $(I - f)(x_*) = 0$ and $(I - g)(x_*) = 0$

$\Rightarrow f(x_*) = x_* = g(x_*)$

Hence x_* is a common fixed point of f and g.

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