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The Existence of Positive Solutions for Four-Point Nonlinear Boundary Value Problem

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Abstract

In this paper, we consider four-point nonlinear boundary value problem

$$(\varphi(u'))' + a(t)f(u) = 0, 0 < t < 1$$

where $\varphi : R \longrightarrow R$ is an increasing homeomorphism and homomorphism and $\varphi(0) = 0$. We show that there exist positive solutions and countably many positive solutions by fixed point theorems in a cone.

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1 Introduction

In this paper, we study the existence of positive solutions of the equation

$$(\varphi(u'))' + a(t)f(u) = 0,\ 0 < t < 1 \tag{1.1}$$

with the following nonlinear four-point boundary value condition

$$u(0) - B_0(u'(\xi)) = 0,\quad u(1) + B_1(u'(\eta)) = 0, \tag{1.2}$$

where $\varphi : R \longrightarrow R$ is an increasing homeomorphism and homomorphism and $\varphi(0) = 0,\ \xi,\ \eta \in (0, 1)$ are constants, and $\xi < \eta$.

In recent paper [1], Eric.R.Kaufmann and Nickolai Kosmatov considered the boundary value problem

$$-u''(t) = a(t)f(u(t)),$$

$$u(0) = u(1) = 0,$$

where $a(t) \in L^p[0,1],\ p \geq 1$, and has countably many singularities in $(0, \frac{1}{2})$. They obtained the existence of countably many positive solutions using Hölder inequality and Krasnosel'skii fixed point theorem for operators on a cone. Moreover, in early works, see, for example, [2-3] and references, focused mainly on problems where the inhomogeneous term was continuous and either sub-linear or suplinear at 0 or ∞. These works established the existence of either one or two positive solutions using a fixed point theorem due to Krasnosel'skii fixed point theorem[7].

When $\eta = 1,\ \xi = 1$, Wang[3] first investigated the nonlinear boundary value problem

$$(\phi_p(u'))' + a(t)f(u) = 0,\ 0 < t < 1$$

with the following nonlinear boundary conditions

$$u(0) - B_0(u'(0)) = 0, \quad u(1) + B_1(u'(1)) = 0,$$

where $\phi_p(u) = |u|^{p-2}u,\ p > 1$, and demonstrated that the problems have at least one positive solution by applying Krasnosel'skii fixed point theorem [5].

Motivated by the results mentioned above, in this paper, we study the existence of positive solutions of the nonlinear boundary value problem (1.1)-(1.2).

A projection $\varphi : R \longrightarrow R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:

(i) if $x \leq y$, then $\varphi(x) \leq \varphi(y), \quad \forall x,\ y \in R$;

(ii) φ is continuous bijection and its inverse mapping is also continuous;

(iii) $\varphi(xy) = \varphi(x)\varphi(y),\ \forall x,\ y \in R$.

The following assumptions are adopted throughout this paper:

(H_1) $\varphi : R \longrightarrow R$ is an increasing homeomorphism and homomorphism;

(H_2) $B_0(v)$ and $B_1(v)$ are both nondecreasing, continuous, odd functions defined on $(-\infty, \infty)$ and at least one of them satisfies the condition that there exists $b > 0$ such that

$$0 \leq B_j(v) \leq bv, \text{ for all } \quad v \geq 0, j = 0,\ 1;$$

(H_3) $f : [0, \infty) \to [0, \infty)$ is continuous;

(H_4) $a(t) \in C((0,1), (0, \infty))$, and $0 < \int_0^1 a(t)dt < \infty$. Moreover, $a(t)$ does not vanish identical on any subinterval of $[0, 1]$.

Let

$$f_0 = \lim_{u \to 0} \frac{f(u)}{\varphi(u)}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{\varphi(u)}.$$

We shall that if $a(t)$ satisfies conditions (H_4) and f satisfies some conditions, then the boundary value problem (1.1)-(1.2) has positive solutions and infinitely many solutions.

2 Preliminary Notes

Definition2.1. *Let E be a Banach space and let $P \subset E$ be a closed nonempty subset of E. Then P is a cone provided :*
(1) $\alpha u + \beta v \in P$ for all $u,\ v \in P$ and for all $\alpha,\ \beta \geq 0$,and
(2) $u, -u \in P$, implies $u \equiv 0$.

Theorem2.1.(Krasnosel'skii) *Let E be a Banach space and P a cone in E. Assume $\Omega_1,\ \Omega_2$ are open subset of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous such that either*
(i) $||Tu|| \leq ||u||$, for $u \in P \cap \partial\Omega_1$, and $||Tu|| \geq ||u||$, for $x \in P \cap \partial\Omega_2$, or
(ii) $||Tu|| \geq ||u||$, for $u \in P \cap \partial\Omega_1$, and $||Tu|| \leq ||u||$, for $x \in P \cap \partial\Omega_2$.
Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma2.1. *When $a(t)$ is continuous in small interval of the (0,1), condition (H_4) holds. Then there exists $\mu \in (0, \frac{1}{2})$, such that*

$$0 < \int_{\mu}^{1-\mu} a(t)dt < \infty, \tag{2.1}$$

and hence the function

$$A(t) = \int_{\mu}^{t} \varphi^{-1}\left(\int_{s}^{t} a(t)dt\right) ds + \int_{t}^{1-\mu} \varphi^{-1}\left(\int_{t}^{s} a(t)dt\right) ds \quad t \in [\mu, 1-\mu]$$

is continuous and positive on $[\mu, 1-\mu]$, furthermore $\min_{t\in[\mu,1-\mu]} A(t) = L > 0$, where φ^{-1} is the inverse function to $\varphi(x)$.

Proof. In fact, suppose condition (H_4) holds. Then we have

$$0 < \int_{\mu}^{1-\mu} a(t)dt < \infty.$$

Let $A_1(t) = \int_t^{1-\mu} \phi_q\left(\int_t^s a(t)dt\right) ds$, and $A_2(t) = \int_\mu^t \phi_q\left(\int_s^t a(t)dt\right) ds$, from condition (H_4), we know that the function $A_1(t)$ is strictly monotone decreasing on $[\mu, 1-\mu]$ and $A_1(1-\mu) = 0$, the function $A_2(t)$ is strictly monotone increasing on $[\mu, 1-\mu]$, and $A_2(\mu) = 0$. So the function $A(t) = A_1(t) + A_2(t)$ is positive on $[\mu, 1-\mu]$, which implies $\min_{t\in[\mu,1-\mu]} A(t) = L > 0$.

Let P be the cone in $C[0,1]$, given by

$$P = \{u \in C[0,1] : u(t) \ \text{ is a nonnegative concave function}\}$$

Lemma2.2[4]. *Let* $u \in P$ *and* $\mu \in (0, \frac{1}{2})$, *then*

$$u(t) \geq \mu \|u\|, \qquad \mu \leq t \leq (1-\mu).$$

Hence, $\|u\| = \sup_{t\in[0,1]} |u(t)|$ *and* $\sigma \in (0,1)$ *such that* $\|u\| = u(\sigma)$.

Now we define an operator $T : P \to P$ by

$$Tu(t) = \begin{cases} B_0 \circ \varphi^{-1}\left(\int_\xi^\sigma a(r) f(u(r)))dr\right) + \int_0^t \varphi^{-1}\left(\int_s^\sigma a(r)f(u(r))dr\right) ds, & 0 \leq t \leq \sigma, \\ \\ B_1 \circ \varphi^{-1}\left(\int_\sigma^\eta a(r)f(u(r))dr\right) + \int_t^1 \varphi^{-1}\left(\int_\sigma^s a(r)f(u(r))dr\right) ds, & \sigma \leq t \leq 1, \end{cases}$$

where φ^{-1} is the inverse function to $\varphi(x)$. Here $\sigma = 0$, if $(Tu)^{'}(0) = 0$; $\sigma = 1$, if $(Tu)^{'}(1) = 0$; otherwise, $\sigma = 0$ is unique solution of the equation

$$g_1(t) = g_2(t) \qquad (2.2),$$

where

$$g_1(t) = B_0 \circ \varphi^{-1}\left(\int_\xi^t a(r)f(u(r)))dr\right) + \int_0^t \varphi^{-1}\left(\int_s^\sigma a(r)f(u(r))dr\right) ds,$$

$$g_2(t) = B_1 \circ \varphi^{-1}\left(\int_t^\eta a(r)f(u(r))dr\right) + \int_t^1 \varphi^{-1}\left(\int_\sigma^s a(r)f(u(r))dr\right) ds.$$

Equation (2.2) has unique solution in $(0,1)$. Because $g_1(t)$ is strictly monotone increasing on $[0,1)$, and $g_1(0) = 0$, $g_2(t)$ is strictly monotone decreasing on $(0,1]$, and $g_2(1) = 0$. Therefore, the operator T is well defined and $(Tu)(\sigma) = \max_{t\in[0,1]}(Tu)(t)$. Since

$$(Tu)^{'} = \begin{cases} \varphi^{-1}\left(\int_t^\sigma a(r)f(u(r))dr\right) ds, & 0 \leq t \leq \sigma, \\ \\ -\varphi^{-1}\left(\int_\sigma^t a(r)f(u(r))dr\right) ds & \sigma \leq t \leq 1, \end{cases}$$

is continuous and nonincreasing on $(0,1)$, and

$$(Tu)^{'}(\sigma) = 0.$$

Moreover $(\varphi(Tu)'(t))^{'} = -a(t)f(u)$, a.e. $t \in (0,1)$. Then, $T(P) \subset P$.

Lemma2.3. $T : P \mapsto P$ *is completely continuous.*

Proof. Because the proof of this Lemma is conventional, we omit it.

3 Main Results

For convenience, we set

$$\mu^* = \frac{2}{L}, \quad \mu_* = \frac{1}{(b+1)\varphi^{-1}(\int_0^1 a(r)dr)}.$$

The main results of this section are the following Theorems.

A The Existence of Positive Solution

Theorem 3.1. *Suppose conditions (H_1), (H_2), (H_3) and (H_4) hold. Assume that f also satisfies*

(A_1) $f(u) \geq \varphi(mr), \quad \mu r \leq u \leq r$;

(A_2) $f(u) \leq \varphi(MR), \quad 0 \leq u \leq R$, where $m \in (\mu^*, \infty)$, $M \in (0, \mu_*)$.

Then, the boundary value problem (1.1)-(1.2) has positive solution u such that $\|u\|$ is between r and R.

Proof. Without loss of generality, we may assume that $r \leq R$ and $0 \leq B_0(v) \leq bv$, for all $v \geq 0$. For any $u \in P$, by Lemma 2.2 we have

$$u(t) \geq \mu||u||, t \in [\mu, 1-\mu]. \tag{3.1}$$

We define two open subsets Ω_1 and Ω_2 of E

$$\Omega_1 = \{u \in P : \|u\| < r\}, \ \Omega_2 = \{u \in P : \|u\| < R\}.$$

For $u \in \partial\Omega_1$, by (3.1) for $t \in [\mu, 1-\mu]$, we have $u(t) \geq \mu||u|| = \mu r$. If ($A_1$) hold, we shall discuss it from three perspectives.

Case1: If $\sigma \in [\mu, 1-\mu]$, from(A_1) and Lemma2.1, we can know

$$\begin{aligned}
&2||Tu|| = 2(Tu)(\sigma)\\
&\geq \textstyle\int_0^\sigma \varphi^{-1}\left(\int_s^\sigma a(r)f(u(r))dr\right)ds\\
&+\textstyle\int_\sigma^1 \varphi^{-1}\left(\int_\sigma^s a(r)f(u(r))dr\right)ds\\
&\geq (mr)\left(\textstyle\int_\mu^\sigma \varphi^{-1}\left(\int_s^\sigma a(r)dr\right)ds\right)\\
&+(mr)\textstyle\int_\sigma^{1-\mu} \varphi^{-1}\left(\int_\sigma^s a(r)dr\right)ds\\
&\geq mrA(\sigma) \geq (mr)L > 2r = 2||u||
\end{aligned}$$

Case2: If $\sigma \in (0,\mu)$, from (A_1) and Lemma2.1, we can know

$$\begin{aligned}
&||Tu|| = (Tu)(\sigma)\\
&\geq \int_\sigma^1 \varphi^{-1}\left(\int_\sigma^s a(r)f(u(r))dr\right)ds\\
&\geq \int_\mu^{1-\mu} \varphi^{-1}\left(\int_\mu^s a(r)f(u(r))dr\right)ds\\
&\geq mr\int_\mu^{1-\mu} \varphi^{-1}\left(\int_\mu^s a(r)dr\right)ds\\
&= mrA(\mu) \geq mL > 2r > r = ||u||.
\end{aligned}$$

Case3: If $\sigma \in (1-\mu,1)$, from(A_1) and Lemma2.1, we can know

$$\begin{aligned}
&||Tu|| = (Tu)(\sigma)\\
&\geq \int_0^\sigma \varphi^{-1}\left(\int_s^\sigma a(r)f(u(r))dr\right)ds\\
&\geq \int_\mu^{1-\mu} \varphi^{-1}\left(\int_s^{1-\mu} a(r)f(u(r))dr\right)ds\\
&\geq mr\int_\mu^{1-\mu} \varphi^{-1}\left(\int_s^{1-\mu} a(r)dr\right)ds\\
&= mrA(1-\mu) \geq mrL > 2r > r = ||u||.
\end{aligned}$$

Thus, in all cases , we have

$$\forall u \in \partial\Omega_1, \quad ||Tu|| \geq ||u||. \tag{3.2}$$

Next, let $u \in \partial\Omega_2$, then $u(s) \leq \|u\| = R$, from condition (A_2), we can know

$$\begin{aligned}
&||Tu|| = (Tu)(\sigma)\\
&\leq B_0 \circ \varphi^{-1}\left(\int_0^1 a(r)f(u(r))dr\right)\\
&+\varphi^{-1}\left(\int_0^1 a(r)f(u(r))dr\right)ds\\
&\leq (1+b)MR\varphi^{-1}(\int_0^1 a(r)dr)\\
&\leq R = ||u||.
\end{aligned}$$

Hence

$$\forall u \in \partial\Omega_2, \quad ||Tu|| \leq ||u||. \tag{3.3}$$

Now $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, from (3.2) (3.3) and Theorem 2.1, we can know operator T has a fixed point $u \in (\overline{\Omega}_2 \setminus \Omega_1)$ such that $r \le \|u\| \le R$.

Theorem3.2 *Suppose conditions (H_1), (H_2), (H_3)and(H_4) hold. Assume that f also satisfies*

(A_3) $f_\infty = \lambda \in \left(\varphi\left(\frac{2\mu^*}{\mu}\right), \infty\right)$;

(A_4) $f_0 = \phi \in \left[0, \varphi\left(\frac{\mu_*}{4}\right)\right)$.

Then, the boundary value problem (1.1)-(1.2) has positive solution u.

Proof. First, by $f_0 = \lim\limits_{u\to 0} \frac{f(u)}{\varphi(u)} = \phi$, for $\epsilon = \varphi\left(\frac{\mu_*}{4}\right) - \phi$, there exists an adequate small positive number ρ, as $0 \le u \le \rho, u \ne 0$, we have

$$f(u) \le (\phi + \epsilon)\varphi(u) \le \varphi\left(\frac{\mu_*}{4}\right)\varphi(2\rho) = \varphi\left(\frac{\mu_*}{2}\rho\right). \tag{3.4}$$

then let $R = \rho,\ M = \frac{\mu_*}{2} \in (0, \mu_*)$, thus by (3.4)

$$f(u) \le \varphi(MR),\ 0 \le u \le R.$$

So condition (A_2) holds.

Next, by condition (A_3), $f_\infty = \lim\limits_{u\to 0} \frac{f(u)}{\varphi(u)} = \lambda \in \left(\varphi\left(\frac{2\mu^*}{\theta}\right), \infty\right)$, then for $\epsilon = \lambda - \varphi\left(\frac{2\mu^*}{\mu}\right)$, there exists an adequate big positive number $r \ne R$, as $u \ge \mu r$, we have

$$f(u) \ge (\lambda - \epsilon)\varphi(u) \ge \varphi\left(\frac{2\mu^*}{\mu}\right)\varphi(\mu r) = \varphi(2\mu^* r). \tag{3.5}$$

Let $m = 2\mu^* > \mu^*$, thus by (3.5) condition (A_1) holds. Therefore by Theorem 3.1 we know that the results of Theorem 3.2 holds. The proof of Theorem 3.2 is complete.

Theorem 3.3. *Suppose conditions (H_1), (H_2), (H_3)and(H_4) hold. Assume that f also satisfies*

(A_5) $f_0 = \phi \in \left(\varphi\left(\frac{2\mu^*}{\mu}\right), \infty\right)$;

(A_6) $f_\infty = \lambda \in \left[0, \varphi\left(\frac{\mu_*}{4}\right)\right)$.

Then, the boundary value problem (1.1)-(1.2) has positive solution u.

Proof. First, by condition (A_5), $f_0 = \phi$, then for $\epsilon = \phi - \varphi\left(\frac{2\mu^*}{\mu}\right)$, there exists an adequate small positive number r, as $0 \le u \le r, u \ne 0$, we have

$$f(u) \ge (\phi - \epsilon)\varphi(u) = \varphi\left(\frac{2\mu^*}{\mu}\right)\varphi(u),$$

thus when $\mu r \le u \le r$, we have

$$f(u) \ge \varphi\left(\frac{2\mu^*}{\mu}\right)\varphi(\mu r) = \varphi(2\mu^* r). \tag{3.6}$$

Let $m = 2\mu^* > \mu^*$, so by (3.6) condition (A_1) holds.

Next, by condition (A_6) $f_\infty = \lambda$, then for $\epsilon = \varphi\left(\frac{\mu_*}{4}\right) - \lambda$, there exists an adequate small positive number $\rho \neq r$, as $u \ge \rho$, we have

$$f(u) \le (\lambda + \epsilon)\varphi(u) \le \varphi\left(\frac{mu_*}{4}\right)\varphi(u). \tag{3.7}$$

If f is non-boundary, by the continuation of f on $[0, \infty)$, then exists constant $R(\neq r) \ge \rho$, and a point $u_0 \in [0, \infty)$ such that $\rho \le u \le R$ and $f(u) \le f(u_0),\ 0 \le u \le R$. Thus, by $\rho \le u_0 \le R$, we know

$$f(u) \le f(u_0) \le \varphi\left(\frac{\mu_*}{4}\right)\varphi(u_0) \le \varphi\left(\frac{\mu_*}{4}R\right).$$

Let $M = \frac{\mu_*}{4} \in (0, \mu_*)$, we have

$$f(u) \le \varphi(MR),\ 0 \le u \le R.$$

If f is boundary, we suppose $f(u) \le \varphi(\overline{M}), u \in [0, \infty)$, there exists an adequate big positive number $R > \frac{4}{\mu_*}\overline{M}$, then let $M = \frac{\mu_*}{4} \in (0, \mu_*)$, we have

$$f(u) \le \varphi(\overline{M}) \le \varphi\left(\frac{\mu_*}{4}R\right) = \varphi(MR),\ 0 \le u \le R.$$

So, condition (A_2) holds. Therefore by Theorem 3.1 we know that the results of Theorem 3.3 holds. The proof of Theorem 3.3 is complete.

B The Existence of Infinitely Many Positive Solutions

Next, we will discuss the existence of infinitely many positive solutions. We suppose that

(H_4') There exists a sequence $\{t_i\}_{i=1}^\infty$ such that $t_{i+1} < t_i,\ t_1 < \frac{1}{2},\ \lim\limits_{i\to\infty} t_i = t^* \ge 0,\ \lim\limits_{t\to t_i} a(t) = \infty,\ \ k = 1, 2, \cdots$, and $0 < \int_0^1 a(t)dt < \infty$. Moreover, $a(t)$ does not vanish identical on any subinterval of $[0, 1]$.

Theorem4.1. *Assume that conditions (H_1), (H_2), $(H_3)$and(H_4')hold. Let $\{\mu_k\}_{k=1}^\infty$ such that $\mu_k \in (t_{k+1}, t_k)(k = 1.2\cdots)$.Let $\{r_k\}_{k=1}^\infty$ and $\{R_k\}_{k=1}^\infty$ such that*

$$R_{k+1} < \mu_k r_k < r_k < mr_k < R_k,\ k = 1, 2, \cdots,$$

Furthermore, for each natural number k, assume that f satisfies

(C_1) *$f(u) \ge \varphi(mr_k)$ for all $u \in [\mu_k r_k, r_k]$;*

(C_2) *$f(u) \le \varphi(MR_k)$ for all $u \in [0, R_k]$.*

Then, the problem (1.1)-(1.2)has infinitely many positive solutions $\{u_i\}_{i=1}^{\infty}$ such that $r_i \le \|u_i\| \le R_i$ for each $i = 1.2\cdots$.

Proof. Without loss of generality, we may assume that $r_k \le R_k$ and $0 \le B_0(v) \le bv$, for all $v \ge 0$. Consider the sequences $\{\Omega_{1,k}\}_{k=1}^{\infty}$ and $\{\Omega_{2,k}\}_{k=1}^{\infty}$ of open subsets of E defined by

$$\Omega_{1,k} = \{u \in K : \|u\| < r_k\},$$
$$\Omega_{2,k} = \{u \in K : \|u\| < R_k\}.$$

Let $\{\mu_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $t_0 < t_{k+1} < \mu_k < t_k < \frac{1}{2}$ for all $k \in N$. Then, for each $k \in N$ and $u \in K$, from Lemma2.2, we have

$$u(t) \ge \mu_k \|u\| \tag{4.1}$$

Now, fix k and let $u \in \partial\Omega_{1,k}$, from (4.1)we have

$$r_k = \|u\| \ge u(s) \ge \mu_k\|u\| = \mu_k r_k, \quad s \in [\mu_k, 1-\mu_k]$$

(Note $[t_1, 1-t_1] \subset [\mu_k, 1-\mu_k]$). In the following, we consider the following three cases.

Case1: If $\sigma \in [t_1, 1-t_1]$, from(C_1) and Lemma2.1, w e can know

$$\begin{aligned}
&2||Tu|| = 2(Tu)(\sigma)\\
&\ge \textstyle\int_0^\sigma \varphi^{-1}\left(\int_s^\sigma a(r)f(u(r))dr\right) ds\\
&+\textstyle\int_\sigma^1 \varphi^{-1}\left(\int_\sigma^s a(r)f(u(r))dr\right) ds\\
&\ge (mr_k)\left(\textstyle\int_{t_1}^\sigma \varphi^{-1}\left(\int_s^\sigma a(r)dr\right) ds\right)\\
&+(mr_k)\left(\textstyle\int_\sigma^{1-t_1} \varphi^{-1}\left(\int_\sigma^s a(r)dr\right) ds\right)\\
&\ge mr_k A(\sigma) \ge (mr_k)L = 2r_k = 2||u||.
\end{aligned}$$

Case2: If $\sigma \in (0, t_1)$, from (C_1) and Lemma2.1, we can know

$$\begin{aligned}
&||Tu|| = (Tu)(\sigma)\\
&\ge \textstyle\int_\sigma^1 \varphi^{-1}\left(\int_\sigma^s a(r)f(u(r))dr\right) ds\\
&\ge \textstyle\int_{t_1}^{1-t_1} \varphi^{-1}\left(\int_{t_1}^s a(r)f(u(r))dr\right) ds\\
&\ge mr_k \textstyle\int_{t_1}^{1-t_1} \varphi^{-1}\left(\int_{t_1}^s a(r)dr\right) ds\\
&= mr_k A(t_1) \ge mr_k L = 2r_k > r_k = ||u||.
\end{aligned}$$

Case3: If $\sigma \in (1-t_1, 1)$, from(C_1) and Lemma2.1, we can know

$$\begin{aligned}
&||Tu|| = (Tu)(\sigma)\\
&\geq \int_0^\sigma \varphi^{-1}\left(\int_s^\sigma a(r)f(u(r))dr\right)ds\\
&\geq \int_{t_1}^{1-t_1} \varphi^{-1}\left(\int_s^{1-t_1} a(r)f(u(r))dr\right)ds\\
&\geq mr_k\int_{t_1}^{1-t_1} \varphi^{-1}\left(\int_s^{1-t_1} a(r)dr\right)ds\\
&= mr_kA(1-t_1) \geq mr_kL = 2r_k > r_k = ||u||.
\end{aligned}$$

Thus, in all cases, we have

$$\forall u \in \partial\Omega_{1,k}, \quad ||Tu|| \geq ||u||. \tag{4.2}$$

Next, let $u \in \partial\Omega_{2,k}$, then $u(s) \leq \|u\| = R_k$, from condition (C_2), we can know

$$\begin{aligned}
&||Tu|| = (Tu)(\sigma)\\
&\leq B_0 \circ \varphi^{-1}\left(\int_0^1 a(r)f(u(r))dr\right)\\
&+\varphi^{-1}\left(\int_0^1 a(r)f(u(r))dr\right)ds\\
&\leq (1+b)MR_k\varphi^{-1}(\int_0^1 a(r)dr)\\
&\leq R_k = ||u||.
\end{aligned}$$

Hence

$$\forall u \in \partial\Omega_{2,k}, \quad ||Tu|| \leq ||u||. \tag{4.3}$$

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Now $0 \in \Omega_{1,k} \subset \overline{\Omega}_{1,k} \subset \Omega_{2,k}$, from (4.2) (4.3) and Theorem 2.1, we can know operator T has a fixed point $u_k \in (\overline{\Omega}_{2,k} \setminus \Omega_{1,k})$ such that $r_k \leq \|u_k\| \leq R_k$. Since $k \in N$ is arbitrary, the proof is complete.

Similar to Theorem 3.2 and 3.3 we have the following Theorems.

Theorem 4.2. *Suppose conditions (H_1), (H_2), $(H_3)$and $(H_4^{'})$ hold. Let $\{\mu_k\}_{k=1}^\infty$ such that $\mu_k \in (t_{k+1}, t_k)(k = 1, 2, \cdots)$. Let $\{r_k\}_{k=1}^\infty$ and $\{R_k\}_{k=1}^\infty$ be such that*

$$R_{k+1} < \mu_k r_k < r_k < mr_k < R_k, \ k = 1, 2, \cdots,$$

For each natural number k we assume that f satisfies
(C_3) $f_\infty = \lambda \in \left(\varphi\left(\frac{2\mu^*}{\mu}\right), \infty\right)$;

(C_4) $f_0 = \phi \in \left[0, \varphi\left(\frac{\mu_*}{4}\right)\right).$

Then, the boundary value problem (1.1)-(1.2) has infinitely many solutions $\{u_k\}_{k=1}^{\infty}$ *such that* $\|u_k\|$ *is between* r_k *and* R_k, $k = 1, 2, \cdots$.

Theorem 4.3. *Suppose conditions (H_1), (H_2), (H_3)and(H_4') hold. Let* $\{\mu_k\}_{k=1}^{\infty}$ *such that* $\mu_k \in (t_{k+1}, t_k)(k = 1, 2, \cdots)$. *Let* $\{r_k\}_{k=1}^{\infty}$ *and* $\{R_k\}_{k=1}^{\infty}$ *be such that*

$$R_{k+1} < \mu_k r_k < r_k < m r_k < R_k,\ k = 1, 2, \cdots,$$

For each natural number k *we assume that* f *satisfies*
(C_5) $f_0 = \phi \in \left(\varphi\left(\frac{2\mu^*}{\mu}\right), \infty\right)$;

(C_6) $f_\infty = \lambda \in \left[0, \varphi\left(\frac{\mu_*}{4}\right)\right).$

Then, the boundary value problem (1.1)-(1.2) has infinitely many solution $\{(u_k, v_k)\}_{k=1}^{\infty}$ *such that* $\|u_k\|$ *is between* r_k *and* R_k, $k = 1, 2, \cdots$.

Remark 1. In fact, we can find a(t) which holds condition (H_4'). Note

$$\Delta = \sqrt{2}(\frac{\pi^2}{3} - \frac{9}{4}),\quad t_0 = \frac{5}{16},\quad t_n = t_0 - \sum_{i=1}^{n-1} \frac{1}{(i+2)^4},\qquad k = 1, 2, \cdots,$$

Consider the function $a(t) : [0, 1] \to (0, +\infty)$, $a(t) = \sum_{n=1}^{\infty} a_n(t)$, $t \in [0, 1]$ where

$$a_n(t) = \begin{cases} \frac{1}{n(n+1)(t_{n+1}+t_n)} & 0 \le t < \frac{t_{n+1}+t_n}{2} \\ \frac{1}{\Delta(t_n - t)^{\frac{1}{2}}} & \frac{t_{n+1}+t_n}{2} \le t < t_n \\ \frac{1}{\Delta(t - t_n)^{\frac{1}{2}}} & t_n \le t \le \frac{t_{n+1}+t_n}{2} \\ \frac{2}{n(n+1)(2 - t_n - t_{n-1})} & \frac{t_{n-1}+t_n}{2} < t \le 1 \end{cases}$$

Obvious $t_1 = \frac{1}{4} < \frac{1}{2}$, $t_n - t_{n+1} = \frac{1}{(n+2)^4}$, $k = 1, 2, \cdots$, (Note$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$)

$$t^* = \lim_{n\to\infty} t_n = \frac{5}{16} - \sum_{i=1}^{\infty} \frac{1}{(i+2)^4} = \frac{21}{16} - \frac{\pi^4}{90} > \frac{1}{5}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ holds , so we can know

$$\begin{aligned}
&\textstyle\sum_{n=1}^{\infty}\int_0^1 a_n(t)dt\\
&=\textstyle\sum_{n=1}^{\infty}\frac{2}{n(n+1)}+\frac{1}{\Delta}\sum_{n=1}^{\infty}\left[\int_{\frac{t_{n+1}+t_n}{2}}^{t_n}\frac{1}{(t_n-t)^{\frac{1}{2}}}dt+\int_{t_n}^{\frac{t_n+t_{n-1}}{2}}\frac{1}{(t-t_n)^{\frac{1}{2}}}dt\right]\\
&=2+\tfrac{\sqrt{2}}{\Delta}\textstyle\sum_{n=1}^{\infty}\left[(t_n-t_{n+1})^{\frac{1}{2}}+(t_{n-1}-t_n)^{\frac{1}{2}}\right]\\
&=2+\tfrac{\sqrt{2}}{\Delta}\textstyle\sum_{n=1}^{\infty}\left[\frac{1}{(n+2)^2}+\frac{1}{(n+1)^2}\right]\\
&=2+\tfrac{\sqrt{2}}{\Delta}\textstyle\sum_{n=1}^{\infty}\left[(\frac{\pi^2}{6}-\frac{5}{4})+(\frac{\pi^2}{6}-1)\right]\\
&=2+\tfrac{\sqrt{2}}{\Delta}[\tfrac{\pi^2}{3}-\tfrac{9}{4}]\\
&=3.
\end{aligned}$$

So

$$\int_0^1 a(t)dt=\int_0^1\sum_{n=1}^{\infty}a_n(t)dt=\sum_{n=1}^{\infty}\int_0^1 a_n(t)dt<\infty$$

Hence a(t) satisfies condition(H_4').

Remark 2. When

$$\varphi(x)=\begin{cases} x^3, & x\le 0,\\ x^2, & x>0,\end{cases}$$

f and a satisfy the conditions (H_1), (H_2), $(H_3)$and(H_4). It is clear that $\varphi: R\longrightarrow R$ is an increasing homeomorphism and homomorphism and $\varphi(0)=0$. Hence we generalize boundary value problem with p-Laplacian operator and the results[4]do not apply to the example.

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