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General Orthogonality in Banach spaces

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Abstract

In this paper, we investigate some properties of the General orthogonality in Banach spaces, and obtain some results on General orthogonality in Banach spaces similar to orthogonality of Hilbert spaces. In this paper we shall consider the relation between this concept in smooth spaces and sense Brikhoff orthogonality.

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1. Introduction

Let B be a real Banach space, B^* its conjugate space, $\|.\|$, $\|.\|_*$ norms in the Banach spaces B, B^*, respectively. Suppose $J: B \to B^*$ is a normalized duality mapping in B determined by the equalities

$$< Jx, x >= \|Jx\|_* \, \|x\| = \|x\|^2.$$

Also J is a one-to-one operator, $J^{-1} = J^*$, where J^{-1} is the inverse operator to J, and $J^* : B^* \to B$ is the normalized duality mapping in B^*. Therefore $J^*J = I_B$ and $JJ^* = I_B^*$. Here $I_B : B \to B$ and $I_B^* : B^* \to B^*$ are identity operators in B and B^*, respectively. For any $x \in B$, there exists a unique element $\phi \in B^*$, such that $\phi = Jx$, consequently $x = J^*\phi$.

Many authors have introduced the concept of orthogonality in different ways [2-6]. In [1] Brikhoff modified the concept of orthogonality. In fact, if $x, y \in B$, x is said to be orthogonal to y and is denoted by $x \perp^{BG} y$ if $\|x\| \leq \|x + \alpha y\|$ for all scalar α. In this note, we shall consider general orthogonality in the Banach spaces.

Definition [**Alber**] **1.1**. *Let B be a real Banach space, $x, y \in B$. x is called general orthogonal to y and write $x \perp^G y$, if $< Jx, y >= 0$ or equivalent, there exists a unique $\phi_x \in B^*$ such that $\phi_x(x) = \|x\|^2$, $\|\phi_x\| = \|x\|$ and $\phi_(y) = 0$.*

At first we state the following lemma which is needed in the proof of the main results.

Lemma [**5**] **1.2**. *Let B be a real Banach space, $x, y \in B$. Then the following two conditions are equivalent:*
(1) $x \perp^{BG} y$
(2)There exists a linear functional Λ on X such that, $\|\Lambda\| = 1$, $\Lambda(x) = \|x\|$ and $\Lambda(y) = 0$.

2. Main Results

In this section we state and prove some characterizations of the general orthogonality in Banach spaces.

Theorem 2.1. *Let B be a real Banach space.Then the following statements are true:*
a) *For all $x \in B$ and all $\alpha > 0$, $\phi_{\alpha x} = \alpha \phi_x$.*
b) *For all $x, y \in B$ and all $\alpha > 0$, if $x \perp^G y$, then $\alpha x \perp^G y$.*
c) *For all $x \in B$, if $x \perp^G x$, then $x = 0$.*
d) *For all $x, y \in B$, if $x \perp^G y$ and $x \neq 0$, then $< x > \cap < y >= \{0\}$.*
e) *For all $x \in B$, $0 \perp^G x$ and $x \perp^G 0$.*

Proof. (a). Suppose $x \in B$ and $\alpha > 0$. Then

$$\alpha\phi_x(\alpha x) = \alpha^2\|x\|^2 = \|\alpha x\|^2 \ , \|\alpha\phi_x\| = \alpha\|\phi_x\| = \alpha\|x\| = \|\alpha x\|,$$

also $\alpha\phi_x(y) = 0$. By uniqueness of $\phi_{\alpha x}$ we have $\phi_{\alpha x} = \alpha\phi_x$.

(b). Proof is a conclusion of (a).

(c). For all $x \in B$, if $x \perp^G x$. Then $\phi_x(x) = 0$ and $\phi_x(x) = \|x\|^2$. Therefore $x = 0$.

(d). If $z \in < x > \cap < y >$, then for scales c_1, c_2, $z = c_1 x = c_2 y$. Hence $\phi_x(z) = 0$, it follows that $\phi_x(c_1 x) = 0$. Therefore $c_1 = 0$ and $z = 0$.

(e). It is trivial. ■

Let B be a Banach space. The element $x \in B$ is called normal element if there exists only one $f \in B^*$ such that $f(x) = \|x\|$ and $\|f\| = 1$.

Theorem 2.2. *Let B be a real Banach space. Then the following statements are true:*
a) *If $x, y \in B$, $x \perp^G y$, then $x \perp^{BG} y$.*
b) *If $x \neq 0 \in B$ is a normal element, $y \in B$ and $x \perp^{BG} y$, then $x \perp^G y$.*

Proof. (a). Suppose $x, y \in B$ and $x \perp^G$ then

$$\begin{aligned} \|x\|^2 &= \phi_x(x) \\ &= \phi_x(x + \alpha y) \\ &\leq \|\phi_x\| \|x + \alpha y\| \\ &= \|x\| \|x + \alpha y\|. \end{aligned}$$

Therefore $\|x\| \leq \|x + \alpha y\|$, that is $x \perp^{BG} y$.

(b). We know that if $x \perp^{BG} y$ and $\alpha > 0$ then $\alpha x \perp^{BG} y$. Therefore $X = \frac{x}{\|x\|} \perp^{BG} y$. Since x is normal by Lemma 1.2, there exists a unique $\phi_X \in B^*$ such that $\phi_X(X) = 1$, $\|\phi_X\| = 1$ and $\phi_X(y) = 0$. From Theorem 2.1, $\phi_X = \frac{1}{\|x\|}\phi_x$. Therefore there exists a unique $\phi_x \in B^*$ such that $\phi_x(x) = \|x\|^2$, $\|\phi_x\| = \|x\|$ and $\phi_(y) = 0$. It follows that $x \perp^G y$. ■

We know that every Hilbert space $(H, < . >)$ is a smooth spaces. Therefore every element $x \in H$ is normal, Hence we have

Corollary 2.3. *Let $(H, < . >)$ be a real Hilbert space, $x, y \in H$. If $x \perp^G y$ then $< x, y >= 0$.* ■

Definition 2.4. *Let B be a real Banach space. The general orthogonality is called G-additivity, if $y \perp^G x$ and $z \perp^G x$, then $y + z \perp^G x$.*

Definition 2.5. *Let B be a real Banach space, $M \subseteq B$ and $x \in B$. Then we say that x is general orthogonal to M and write $x \perp^G M$ if there exists a unique $\phi_x \in B^*$ such that $\phi_x(x) = \|x\|^2$, $\|\phi_x\| = \|x\|$ and for all $y \in M$ $\phi_(y) = 0$. The element $y_0 \in M$ is G-best approximation of $x \in B$ if and only if $x - y_0 \perp^G M$.*

Corollary 2.6. *Let B be a real Banach space, $x \in B$. If $y_0 \in M$ is G-best approximation of $x \in B$. Then y_0 is a best approximation of x.*

Theorem 2.7. *Let B be a real Banach space, $x \in B$ and $M \subseteq B$. If the general orthogonality is G-additivity, Then there exist a unique $y_0 \in M$ such that $x - y_0 \perp^G M$.*

Proof. Suppose $y_1, y_2 \in M$ such that for all $i = 1, 2$, $x - y_i \perp^G M$. Therefore for all $y \in M$ and all $i = 1, 2$, $x - y_i \perp^G y$. Since orthogonality is G-additivity. It follows that $y_2 - y_1 = (x - y_1) - (x - y_2) \perp^G y$. Put $y = y_1 - y_2$, then $y_1 - y_2 \perp^G y_1 - y_2$. From Theorem 2.1, $y_1 = y_2$. ■

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