

Fuzzy Calculus for Strong Limiting Subdifferential in Banach Spaces

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Abstract

We give in this paper some fuzzy calculus results related to strong limiting subdifferential in Banach spaces.

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1 Introduction

The strong limiting subdifferential was first introduced by S.Lahrech, J. Hlal, A. Jaddar, A.Ouahab and A. Mbarki in [1] in order to characterize the extrema of pseudoconvex functions via this a new kind of limiting subdifferential.

We establish in this paper some interesting fuzzy calculus rules related to strong limiting subdifferential for extended-real-valued functions defined on Banach spaces.

The rest of this paper is organized as follows. Section 2 contains some definitions and preliminary material, while in section 3, we establish the result related to sum rule and in section 4 we prove some other calculus results: chain rule, products and quotients rules.

2 Basic definitions and properties

In this section, we recall several definitions and results necessary in the sequel. Throughout this paper, X and Y denote two Banach spaces and X^* the topological dual of X equipped with the weak-star topology.

By $B_\rho(x)$ we denote the open ball centered at x with radius ρ ($\rho > 0$).

For any extended-real-valued function $f : X \rightarrow \bar{R}$, the domain of f is defined by:

$$\text{dom}f = \{x \in X / |f(x)| < +\infty\}.$$

The symbol $x \xrightarrow{f} \bar{x}$ (respectively, $\xrightarrow{\omega^*}$) means that $x \rightarrow \bar{x}$ with $f(x) \rightarrow f(\bar{x})$ (respectively, the convergence for the weak-star topology of X^*).

Definition 2.1 [8] *Let $f : X \rightarrow R \cup \{+\infty\}$ be an extended-real-valued function, and let $\bar{x} \in \text{dom}f$.*

The set

$$\hat{\partial}_\varepsilon f(x) = \{x^* \in X^* / \liminf_{u \rightarrow x} \frac{f(u) - f(x) - \langle x^*, u - x \rangle}{\|u - x\|} \geq -\varepsilon\}, \quad \varepsilon \geq 0 \quad (1)$$

is called the ε -Fréchet subdifferential of f at x .

When $\varepsilon = 0$, then the set (1) is called the presubdifferential or Fréchet subdifferential of f at x and is denoted by $\hat{\partial}f(x)$

Definition 2.2 *Let $f : X \rightarrow R \cup \{+\infty\}$ be an extended-real-valued function, and let $\bar{x} \in \text{dom}f$. Assume that f is l.s.c around \bar{x} .*

We define the strong limiting subdifferential $\check{\partial}f(\bar{x})$ by:

$$\check{\partial}f(\bar{x}) = \limsup_{\rho \searrow 0, \varepsilon \searrow 0, x \xrightarrow{f, \rho} \bar{x}} \hat{\partial}_\varepsilon f(x), \quad (2)$$

where $\limsup_{\rho \searrow 0, \varepsilon \searrow 0, x \xrightarrow{f} \bar{x}} \hat{\partial}_\varepsilon f(x)$ denotes the strong sequential Kuratowski-Painlevé upper limit, that is:

$$\limsup_{\rho \searrow 0, \varepsilon \searrow 0, x \xrightarrow{f} \bar{x}} \hat{\partial}_\varepsilon f(x) = \{x^* \in X^* / \exists \varepsilon_k \downarrow 0, \exists \rho_k \downarrow 0, \exists x_k \xrightarrow{f} \bar{x}, \exists x_k^* \xrightarrow{\omega^*} x^* \text{ such that}$$

$(\rho_k)_k$ is a nonincreasing sequence satisfying: $\rho_k > 0$ and $\forall (k_1, k_2)$ large enough,

there is a positive integer $k_3 \geq k_2$ such that

$$\forall x' \in B_{\rho_{k_1}}(x_{k_3}) \quad f(x') - f(x_{k_3}) - \langle x_{k_3}^*, x' - x_{k_3} \rangle \geq -2\varepsilon_{k_3} \|x' - x_{k_3}\|$$

$$\text{and } \|\bar{x} - x_{k_3}\| < \frac{\rho_{k_3}}{2} \}.$$

3 Sum rule

In this section, we establish some results related to sum rule for strong limiting subdifferential introduced in section 2.

Definition 3.1 Let $f : X \rightarrow R \cup \{+\infty\}$ be an extended-real-valued function, and let $x \in \text{dom} f$. Assume that f is l.s.c around x . The set

$$\hat{\partial}_s f(x) = \{x^* \in X^* / \forall x_n \rightarrow x \exists \delta_n \rightarrow 0 (\delta_n > 0) \exists \delta > 0 \exists n_0 \in N$$

$$\forall n \geq n_0 \forall z \in B_\delta(x)$$

$$f(x_n) - f(z) - \langle x^*, x_n - z \rangle \geq -\delta_n \|z - x_n\| \}$$

is called the strict Fréchet sequential subdifferential of f at x .

If $\hat{\partial}_s f(x) \neq \emptyset$, then we say that f is strictly sequentially Fréchet subdifferentiable at x .

Theorem 3.2 Let $\varphi : X \rightarrow R, \psi : X \rightarrow R$ be two l.s.c real functions around $\bar{x} \in X$. Assume that φ is sequentially strictly Fréchet subdifferentiable at \bar{x} . Then

$$\check{\partial}(\varphi + \psi)(\bar{x}) \subset \hat{\partial}_s \varphi(\bar{x}) + \check{\partial} \psi(\bar{x}).$$

Proof. Let $x^* \in \check{\partial}(\varphi + \psi)(\bar{x})$. Then, there are sequences $\rho_k \downarrow 0, \varepsilon_k \downarrow 0, x_k \xrightarrow{\varphi + \psi} \bar{x}, x_k^* \xrightarrow{\omega^*} x^*$ such that $(\rho_k)_k$ is a nonincreasing sequence satisfying: $\rho_k > 0 (\forall k), \forall \delta > 0 \forall (k_1, k_2)$ large enough, there is a positive integer $k_3 \geq \max(k_1, k_2)$ such that

$$\forall x \in B_{\rho_{k_1}}(x_{k_3}) \quad (\varphi + \psi)(x) - (\varphi + \psi)(x_{k_3}) - \langle x_{k_3}^*, x - x_{k_3} \rangle \geq -2\varepsilon_{k_3} \|x - x_{k_3}\|, \tag{3}$$

and $\|\bar{x} - x_{k_3}\| < \frac{\rho_{k_3}}{2}$. Since φ is sequentially strictly Fréchet subdifferentiable at \bar{x} , then there is $x_1^* \in \hat{\partial}_s \varphi(\bar{x})$. Consequently,

$$\exists \delta_n \rightarrow 0 \ (\delta_n > 0) \quad \exists \delta > 0 \quad \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 \quad \forall z \in B_\delta(\bar{x})$$

$$\varphi(x_n) - \varphi(z) - \langle x_1^*, x_n - z \rangle \geq -\delta_n \|z - x_n\|.$$

Let k_1, k_2 be two large positive integers. Then, there is a positive integer $k_3 \geq \max(k_1, k_2, n_0)$ such that $\rho_{k_1} + \frac{\rho_{k_3}}{2} \leq \frac{\delta}{2}$ and

$$\forall x \in B_{\rho_{k_1}}(x_{k_3}) \quad (\varphi + \psi)(x) - (\varphi + \psi)(x_{k_3}) - \langle x_{k_3}^*, x - x_{k_3} \rangle \geq -2\varepsilon_{k_3} \|x - x_{k_3}\|$$

and $\|\bar{x} - x_{k_3}\| < \frac{\rho_{k_3}}{2}$. Observe that if $x \in B_{\rho_{k_1}}(x_{k_3})$, then $x \in B_\delta(\bar{x})$. Therefore, we deduce that

$$\forall x \in B_{\rho_{k_1}}(x_{k_3}) \quad \varphi(x_{k_3}) - \varphi(x) - \langle x_1^*, x_{k_3} - x \rangle \geq -\delta_{k_3} \|x - x_{k_3}\|. \quad (4)$$

Consequently, by virtue of (3) and (4), we obtain

$$\forall x \in B_{\rho_{k_1}}(x_{k_3}) \quad \psi(x) - \psi(x_{k_3}) - \langle x_{k_3}^* - x_1^*, x - x_{k_3} \rangle \geq -(2\varepsilon_{k_3} + \delta_{k_3}) \|x - x_{k_3}\|.$$

On the other hand, we have $(\varphi + \psi)(x_k) \rightarrow (\varphi + \psi)(\bar{x})$, $x_k \rightarrow \bar{x}$, φ and ψ are l.s.c. Hence, $\psi(x_{k_v}) \rightarrow \psi(\bar{x})$. Consequently, $x^* - x_1^* \in \check{\partial}\psi(\bar{x})$. Thus, $\check{\partial}(\varphi + \psi)(\bar{x}) \subset \hat{\partial}_s \varphi(\bar{x}) + \check{\partial}\psi(\bar{x})$.

4 Chain rule and other fuzzy calculus rules

Analogously to section 3, we give other fuzzy calculus rules for our strong limiting subdifferential.

Definition 4.1 *Let $\Phi : X \rightarrow Y$ be a continuous single-valued mapping and let $\varphi : X \times Y \rightarrow \mathbb{R}$ be a real-valued function. Set $\bar{y} = \Phi(\bar{x})$. Assume that $\varphi(\bar{x}, \cdot)$ is Fréchet differentiable at \bar{y} .*

We say that φ is Φ -sequentially strictly Fréchet differentiable at \bar{x} if

$$\forall \rho_k \downarrow 0, \forall \varepsilon_k \downarrow 0, \forall x_k \xrightarrow{\varphi \circ \Phi} \bar{x}, \forall k_1, k_3 \text{ such that } k_3 \geq k_1$$

the following implication holds:

$$\begin{aligned} [\exists x_1^* \in \hat{\partial}_s \varphi(\cdot, \bar{y})(\bar{x}) \text{ such that } \forall x \in B_{\rho_{k_1}}(x_{k_3}) \quad \varphi(x_{k_3}, \bar{y}) - \varphi(x, \bar{y}) - \langle x_1^*, x_{k_3} - x \rangle \\ \geq -\varepsilon_{k_3} \|x - x_{k_3}\|] \Rightarrow \end{aligned}$$

$$\begin{aligned} \exists \gamma_k \downarrow 0 \text{ such that } \varphi(x_{k_3}, \Phi(x_{k_3})) - \varphi(x, \Phi(x)) - \langle x_1^*, x_{k_3} - x \rangle - \langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi(x_{k_3}) - \Phi(x) \rangle \\ \geq -\gamma_{k_3} (\|x_{k_3} - x\| + \|\Phi(x_{k_3}) - \Phi(x)\|) \quad \forall x \in B_{\rho_{k_1}}(x_{k_3}). \end{aligned}$$

Let us start with the theorem of chain rule.

Theorem 4.2 *Let $\Phi : X \rightarrow Y$ be a continuous single-valued mapping such that $\Phi : X \rightarrow Y$ is locally Lipschitzian around $\bar{x} \in X$. Set $\bar{y} = \Phi(\bar{x})$. Let $\varphi : X \times Y \rightarrow R$ be a Φ -sequentially strictly Fréchet differentiable mapping at \bar{x} . Assume that φ is l.s.c around $(\bar{x}, \Phi(\bar{x}))$. Suppose also that $\varphi(\cdot, \Phi(\bar{x}))$ is sequentially strictly Fréchet subdifferentiable at \bar{x} . Then*

$$\check{\partial}(\varphi \circ \Phi)(\bar{x}) \subset \hat{\partial}_s \varphi(\cdot, \bar{y})(\bar{x}) + \check{\partial} \langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi \rangle(\bar{x}),$$

where $\varphi \circ \Phi$ is the function acting from X into R defined by:

$$(\varphi \circ \Phi)(x) = \varphi(x, \Phi(x)).$$

Proof. Under our hypothesis, we can easily see that $\varphi \circ \Phi$ is l.s.c. around \bar{x} and $\langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi \rangle$ is continuous around \bar{x} . Therefore, we can talk about $\check{\partial}(\varphi \circ \Phi)(\bar{x})$ and $\check{\partial} \langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi \rangle(\bar{x})$.

Denote by l the Lipschitz constant of Ψ at \bar{x} . Then there exists $\delta > 0$ such that

$$\forall x, u \in B_\delta(\bar{x}) \quad \|\Phi(u) - \Phi(x)\| \leq l \|u - x\|.$$

Let $x^* \in \check{\partial}(\varphi \circ \Phi)(\bar{x})$. Then, there are sequences $\rho_k \downarrow 0$, $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\varphi \circ \Phi} \bar{x}$, $x_k^* \xrightarrow{\omega^*} x^*$ such that $(\rho_k)_k$ is a nonincreasing sequence satisfying: $\rho_k > 0 (\forall k)$, $\forall \delta > 0 \forall (k_1, k_2)$ large enough, there is a positive integer $k_3 \geq \max(k_1, k_2)$ such that

$$\forall x \in B_{\rho_{k_1}}(x_{k_3}) \quad (\varphi \circ \Phi)(x) - (\varphi \circ \Phi)(x_{k_3}) - \langle x_{k_3}^*, x - x_{k_3} \rangle \geq -2\varepsilon_{k_3} \|x - x_{k_3}\|, \tag{5}$$

and $\|\bar{x} - x_{k_3}\| < \frac{\rho_{k_3}}{2}$. Since $\varphi(\cdot, \bar{y})$ is sequentially strictly Fréchet subdifferentiable at \bar{x} , then there is $x_1^* \in \hat{\partial}_s \varphi(\cdot, \bar{y})(\bar{x})$. Consequently,

$$\exists \delta_n \rightarrow 0 \ (\delta_n > 0) \quad \exists \delta > 0 \quad \exists n_0 \in N \quad \forall n \geq n_0 \quad \forall z \in B_\delta(\bar{x})$$

$$\varphi(x_n, \bar{y}) - \varphi(z, \bar{y}) - \langle x_1^*, x_n - z \rangle \geq -\delta_n \|z - x_n\|.$$

Let k_1, k_2 be two large positive integers. Then, there is a positive integer $k_3 \geq \max(k_1, k_2, n_0)$ such that $\rho_{k_1} + \frac{\rho_{k_3}}{2} \leq \frac{\delta}{2}$,

$$\forall x \in B_{\rho_{k_1}}(x_{k_3}) \quad (\varphi \circ \Phi)(x) - (\varphi \circ \Phi)(x_{k_3}) - \langle x_{k_3}^*, x - x_{k_3} \rangle \geq -2\varepsilon_{k_3} \|x - x_{k_3}\| \tag{6}$$

and $\|\bar{x} - x_{k_3}\| < \frac{\rho_{k_3}}{2}$. Observe that if $x \in B_{\rho_{k_1}}(x_{k_3})$, then $x \in B_\delta(\bar{x})$. Therefore, we deduce that

$$\forall x \in B_{\rho_{k_1}}(x_{k_3}) \quad \varphi(x_{k_3}, \bar{y}) - \varphi(x, \bar{y}) - \langle x_1^*, x_{k_3} - x \rangle \geq -\delta_{k_3} \|x - x_{k_3}\|.$$

Since φ is Φ -sequentially strictly Fréchet differentiable at \bar{x} , then

$$\exists \gamma_k \downarrow 0 \text{ such that } \varphi(x_{k_3}, \Phi(x_{k_3})) - \varphi(x, \Phi(x)) - \langle x_1^*, x_{k_3} - x \rangle - \langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi(x_{k_3}) - \Phi(x) \rangle$$

$$\geq -\gamma_{k_3}(\|x_{k_3} - x\| + \|\Phi(x_{k_3}) - \Phi(x)\|) \quad \forall x \in B_{\rho_{k_1}}(x_{k_3}).$$

Hence, $\forall x \in B_{\rho_{k_1}}(x_{k_3})$

$$\begin{aligned} & \varphi(x_{k_3}, \Phi(x_{k_3})) - \varphi(x, \Phi(x)) - \langle x_1^*, x_{k_3} - x \rangle - \langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi(x_{k_3}) - \Phi(x) \rangle \geq \\ & -\gamma_{k_3}(\|x_{k_3} - x\| + l\|x_{k_3} - x\|). \end{aligned} \tag{7}$$

By virtue of (6) and (7), we obtain: $\forall x \in B_{\rho_{k_1}}(x_{k_3})$

$$\begin{aligned} & \langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi(x) \rangle - \langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi(x_{k_3}) \rangle - \langle x_{k_3}^* - x_1^*, x - x_{k_3} \rangle \\ & \geq -[2\varepsilon_{k_3} + \gamma_{k_3}(l + 1)] \|x - x_{k_3}\|. \end{aligned}$$

Consequently,

$$x^* - x_1^* \in \check{\partial} \langle \nabla_y \varphi(\bar{x}, \bar{y}), \Phi \rangle(\bar{x}).$$

Thus, we achieve the proof.

At the end of this section we give some additional calculus formulas for the strong limiting subdifferential.

The first one is the product rules involving locally Lipschitzian functions.

Theorem 4.3 *Let $\varphi_i : X \rightarrow R, i = 1, 2$ be two locally Lipschitzian functions around $\bar{x} \in X$. Assume that all φ_i are continuous. Assume that*

$$\forall \rho_k \downarrow 0, \forall \varepsilon_k \downarrow 0, \forall x_k \rightarrow \bar{x}, \forall k_1, k_3 \text{ such that } k_3 \geq k_1$$

$$\exists \gamma_k \downarrow 0 \text{ such that } \forall x \in B_{\rho_{k_1}}(x_{k_3})$$

$$\begin{aligned} & \varphi_1(x_{k_3})\varphi_2(x_{k_3}) - \varphi_1(x)\varphi_2(x) - \varphi_2(\bar{x})(\varphi_1(x_{k_3}) - \varphi_1(x)) - \varphi_1(\bar{x})(\varphi_2(x_{k_3}) - \varphi_2(x)) \\ & \geq -\gamma_{k_3}(\|x_{k_3} - x\| + |\varphi_1(x_{k_3}) - \varphi_1(x)| + |\varphi_2(x_{k_3}) - \varphi_2(x)|). \end{aligned}$$

Then

$$\check{\partial}(\varphi_1 \cdot \varphi_2)(\bar{x}) \subset \check{\partial}(\varphi_2(\bar{x})\varphi_1 + \varphi_1(\bar{x})\varphi_2)(\bar{x}). \tag{8}$$

Proof. Consider the smooth function $\varphi : X \times R^2 \rightarrow R$ and the locally Lipschitzian mapping (around \bar{x}) $\Phi : X \rightarrow R^2$ defined by:

$$\Phi(x) = (\varphi_1(x), \varphi_2(x)), \quad \varphi(x, y_1, y_2) = y_1 \cdot y_2.$$

Using the same argument as in the proof of theorem 6 and taking into account that $0 \in \hat{\partial}_s \varphi(\cdot, \bar{y})(\bar{x})$, we deduce the result.

Theorem 4.4 *Let $\varphi_i : X \rightarrow R, i = 1, 2$ be two locally Lipschitzian functions around $\bar{x} \in X$. Assume that all φ_i are continuous and $\varphi_2(\bar{x}) \neq 0$. Suppose also that*

$$\forall \rho_k \downarrow 0, \forall \varepsilon_k \downarrow 0, \forall x_k \rightarrow \bar{x}, \forall k_1, k_3 \text{ such that } k_3 \geq k_1$$

$$\exists \gamma_k \downarrow 0 \text{ such that } \forall x \in B_{\rho_{k_1}}(x_{k_3})$$

$$\begin{aligned} & \frac{\varphi_1(x_{k_3})}{\varphi_2(x_{k_3})} - \frac{\varphi_1(x)}{\varphi_2(x)} - \frac{1}{\varphi_2(\bar{x})}(\varphi_1(x_{k_3}) - \varphi_1(x)) - \varphi_1(\bar{x})\left(\frac{1}{\varphi_2(x_{k_3})} - \frac{1}{\varphi_2(x)}\right) \\ & \geq -\gamma_{k_3}(\|x_{k_3} - x\| + |\varphi_1(x_{k_3}) - \varphi_1(x)| + \left|\frac{1}{\varphi_2(x_{k_3})} - \frac{1}{\varphi_2(x)}\right|). \end{aligned}$$

Then

$$\check{\partial}\left(\frac{\varphi_1}{\varphi_2}\right)(\bar{x}) \subset \frac{\check{\partial}(\varphi_2(\bar{x})\varphi_1 - \varphi_1(\bar{x})\varphi_2)(\bar{x})}{[\varphi_2(\bar{x})]^2}. \tag{9}$$

Proof. Consider the smooth function $\varphi : X \times R^2 \rightarrow R$ and the locally Lipschitzian mapping (around \bar{x}) $\Phi : R^2 \rightarrow X$ defined by:

$$\Phi(x) = (\varphi_1(x), \varphi_2(x)), \varphi(x, y_1, y_2) = \frac{y_1}{y_2}.$$

Using the same argument as in the proof of theorem 6 and taking into account that $0 \in \hat{\partial}_s \varphi(\cdot, \bar{y})(\bar{x})$, we deduce the result.

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