

# A Combined Walsh Function and Sumudu Transform for Solving the Two-dimensional Neutron Transport Equation

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**Abstract.** In this paper the Walsh function and the Sumudu transform are combined to solve analytically the neutron transport equation in two-dimensional case. The procedure is based on the expansion of the angular flux in terms of the Walsh function the resulting system of linear differential equation is solved analytically using the Sumudu Transform technique.

**Mathematics Subject Classification:** 65D32, 82D75; 40A10

**Keywords:** linear transport equation, isotropic scattering, Wash function

## 1. INTRODUCTION

The neutron transport equation is a linear case of the Boltzmann equation with wide applications in physics and engineering.

As is well known, the study of a given transport equation is a quite important and interesting in transport theory. Various methods have been developed to investigate, and special attention has been given to the task of searching methods that generate accurate results to transport problems in the context of deterministic methods based on analytical procedures, for the multidimensional transport problems, one of the effective methods to treat linear transport equation is the spectral method [30, 29, 31], etc..., whose basic goals is to find

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exact solution for approximations of the transport equation, several approaches have been suggested. Among them, the method proposed by Chandrasekhar [13] solves analytically the discrete equations, ( $S_N$  equations), the spherical harmonics method [16] expands the angular flux in Legendre polynomials, the  $F_N$  method [20] transforms the transport equation into an integral equation. The integral transform technique like the Laplace, Fourier and Bessel also have been applied to solve the transport equation in semi-infinite domain [18, 19], the  $SGF$  method [5, 6], is a numerical nodal method that generates numerical solution for the  $S_N$  equations in slab geometry that is completely free of spatial truncation error. The  $LTS_N$  method [34] solve analytically the  $S_N$  equations employing the Laplace Transform technique in the spatial variable (finite domain). Recently, following the idea encompassed by the  $LTS_N$  method, we have derived a generic method, prevailing the analyticity, for solving one-dimensional approximation that transform the transport equation into a set differential equations.

The version of this generic method are known as  $LTS_N$  [4],  $LTP_N$  [35],  $LTW_N$  [9],  $LTCh_N$  [10],  $LTA_N$  [11],  $LTD_N$  [7].

The analytical character of this solution, in the sense that no approximation is made along its derivation, constitutes its main feature. The idea encompassed is threefold: application of the Laplace Transform to the set of ordinary equations resulting from the approximation, analytical solution of the resulting linear system depending on the complex parameter  $s$  and inversion of the transformed angular flux by the Heaviside expansion technique.

We remark that the second step was accomplished by the application of the procedures that we shall describe further ahead. For the  $LTS_N$  approach, exploiting the structure of the corresponding matrix, the inversion was performed by employing the definition of matrix inversion [4]. On the other hand, for the remaining approaches, the matrix inversion was performed by the *Trzaska's* method [33].

The series expansions method has been largely used in the solution of the differential equation. In particular, Legendre polynomials [16] the Sumudu transform [25] and the Walsh function [9, 32] expansion have been employed to solve the one-dimensional linear transport the Chebyshev polynomials have been employed to solve the two-dimensional linear transport [1, 26] and for three dimensional case [27, 28].

According to Gottlieb [21], spectral method involve representation the solution to a problem as a truncated series of known functions of the independent variables, of course there exist other method to determine the coefficients of expansion, but in regard to that, we should prefer to use orthogonal basis such that those coefficients could be determined by orthogonality properties. Thereby, the orthogonal functions and polynomial series have received considerable attention in dealing with various problem. The main characteristic of this technique is that reduces this problems to those of solving a system

of algebraic equations, thus greatly simplifying the problem and making it computational plausible.

In the present paper we present a new approximation for the two-dimensional transport equation, using Walsh function combined with the Sumudu transform. The approach is based on expansion of the angular flux in a truncated series of Walsh function in the angular variable. By replacing this development in the transport equation, this which will result a first-order linear differential system is solved for the spatial function coefficients by application of the Sumudu transform technique [8].

The inversion of the transformed coefficients is obtained using *Trzaska's* method [33] and the Heaviside expansion technique. In our knowledge, the combination of the Walsh function and the Sumudu Transform to solve the two-dimensional transport equation, in this setting, is not considered in the literature.

## 2. WALSH FUNCTION

The Walsh functions have many properties similar to those of the trigonometric functions. For example they form a complete, total collection of functions with respect to the space of square Lebesgue integrable functions. However, they are simpler in structure to the trigonometric functions because they take only the values 1 and  $-1$ . They may be expressed as linear combinations of the Haar functions [22], so many proofs about the Haar functions carry over to the Walsh system easily. Moreover, the Walsh functions are Haar wavelet packets; see [37]. For a good account of the properties of the Haar wavelets and other wavelets. We use the ordering of the Walsh functions due to Paley [24]. Any function  $f \in L^2 [0, 1]$  can be expanded as a series of Walsh functions

$$f(x) = \sum_{i=0}^{\infty} c_i W_i(x) \quad \text{where } c_i = \int_0^1 f(x) W_i(x). \quad (2.1)$$

Fine [17] discovered an important property of the Walsh Fourier series: the  $m = 2^n$  th partial sum of the Walsh series of a function  $f$  is piece-wise constant, equal to the  $L^1$  mean of  $f$ , on each subinterval  $[(i-1)/m, i/m]$ . For this reason, Walsh series in applications are always truncated to  $m = 2^n$  terms. In this case, the coefficients  $c_i$  of the Walsh (-Fourier) series are given by

$$c_i = \sum_{j=0}^{m-1} \frac{1}{m} W_{ij} f_j, \quad (2.2)$$

where  $f_j$  is the average value of the function  $f(x)$  in the  $j$ th interval of width  $1/m$  in the interval  $(0, 1)$ , and  $W_{ij}$  is the value of the  $i$ th Walsh function in the  $j$ th subinterval. The order  $m$  Walsh matrix,  $\mathcal{W}_m$ , has elements  $W_{ij}$ .

Let  $f(x)$  have a Walsh series with coefficients  $c_i$  and its integral from 0 to  $x$  have a Walsh series with coefficients of  $b_i : \int_0^x f(t) dt = \sum_{i=0}^{\infty} b_i W_i(x)$ . If we

truncate to  $m = 2^n$  terms and use the obvious vector notation, then integration is performed by matrix multiplication  $\mathbf{b} = P_m^T \mathbf{c}$  where

$$P_m^T = \begin{bmatrix} P_{m/2} & \frac{1}{2m} I_{m/2} \\ -\frac{1}{2m} I_{m/2} & O_{m/2} \end{bmatrix}, \quad P_2^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix}, \quad (2.3)$$

and  $I_m$  is the unit matrix,  $O_m$  is the zero matrix (of order  $m$ ), see [15].

### 3. SUMUDU TRANSFORM

The Sumudu Transform, was previously firmly established by many authors, as the theoretical dual to the Laplace Transform, where from the Laplace-Sumudu Duality (**LSD**). In fact, due to its units and scale preserving properties, in many instances, the Sumudu may be preferred to its dual for solving problems in engineering mathematics, without leaving the initial argument domain. Many fundamental Sumudu properties were presented in the literature, by several authors [8, 36].

The Sumudu Transform is defined by,

$$G(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t} dt, \quad u \in (-\tau_1, \tau_2), \quad (3.1)$$

over the set of functions,

$$A = \left\{ f(t) / \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{t}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}. \quad (3.2)$$

Hence,  $G(u)$ , is referred to as *the Sumudu* of  $f(t)$ . For further details we refer to [8].

### 4. THE TWO-DIMENSIONAL SPECTRAL SOLUTION

Consider the two-dimensional linear, steady state, transport equation given by

$$\begin{aligned} & \mu \frac{\partial}{\partial x} \Psi(x, y, \mu, \phi) + \sqrt{1 - \mu^2} \cos \phi \frac{\partial}{\partial y} \Psi(x, y, \mu, \phi) + \sigma_t \Psi(x, y, \mu, \phi) \\ & = \int_{-1}^1 \int_0^{2\pi} \sigma_s(\mu', \phi' \rightarrow \mu, \phi) \Psi(x, y, \mu', \phi') d\phi' d\mu' + S(x, y, \mu, \phi) \end{aligned} \quad (4.1)$$

in the domain  $\Omega = \{\mathbf{x} := (x, y): 0 \leq x \leq 1, -1 \leq y \leq 1\}$  and the direction in  $D = \{(\mu, \theta) : -1 \leq \mu \leq 1, 0 \leq \theta \leq 2\pi\}$ . Here  $\Psi(\mathbf{x}, \mu, \phi)$  is the angular flux,  $\sigma_t$  and  $\sigma_s$  denote the total and the differential cross section, respectively,  $\sigma_s(\mu', \phi' \rightarrow \mu, \phi)$  describes the scattering from an assumed pre-collision angular coordinates  $(\mu', \theta')$  to a post-collision coordinates  $(\mu, \theta)$  and  $S$  is the source term. See [23] for the details.

Note that, in the case of one-speed neutron transport equation; taking the angular variable in a disc, this problem would corresponds to a three dimensional case with all functions being constant in the azimuthal direction of the  $z$  variable. In this way the actual spatial domain may be assumed to be a

cylinder with the cross-section  $\Omega$  and the axial symmetry in  $z$ . Then  $D$  will correspond to the projection of the points on the unit sphere (the "speed") onto the unit disc (which coincides with  $D$ .) See, [2] for the details.

Given the functions  $f_1(y, \mu, \phi)$  and  $f_2(x, \mu, \phi)$ , describing the incident flux, we seek for a solution of (4.1) subject to the following boundary conditions:

For  $0 \leq \theta \leq 2\pi$ , let

$$\Psi(x, y, \mu, \theta) = \begin{cases} f_1(y, \mu, \phi), & \text{for } x = 0; \quad 0 < \mu \leq 1, \\ 0, & \text{for } x = 1; \quad -1 \leq \mu < 0. \end{cases} \quad (4.2)$$

For  $-1 < \mu < 1$ , let

$$\Psi(x, y = \pm 1, \mu, \theta) = \begin{cases} f_2(y, \mu, \phi), & \text{for } y = -1, \quad 0 < \cos \theta \leq 1, \\ 0, & \text{for } y = 1, \quad -1 \leq \cos \theta < 0. \end{cases} \quad (4.3)$$

**Theorem 4.1.** *Consider the integro-differential equation (4.1) under the boundary conditions (4.2) and (4.3), then the function  $\Psi(x, y, \mu, \theta)$  satisfy the following first-order linear differential equation system for the spatial component  $\Psi_k(x, \mu, \theta)$*

$$\begin{aligned} & \mu \frac{\partial}{\partial x} \Psi_k(x, \mu, \theta) + \sigma_t \Psi_k(x, \mu, \theta) \\ &= \int_{-1}^1 \int_0^{2\pi} \sigma_s(\mu', \phi' \rightarrow \mu, \phi) \Psi_k(x, \mu', \phi') d\theta' d\mu' + G_k(x, \mu, \theta) \end{aligned}$$

where

$$G_k(x, \mu, \theta) = S_k(x, \mu, \theta) - \sqrt{1 - \mu^2} \cos \theta \sum_{i=k+1}^I A_i^k \Psi_k(x, \mu, \theta)$$

and

$$\begin{aligned} A_i^k &= \frac{2}{\pi} \int_{-1}^1 \frac{d}{dy} (T_i(y)) \frac{T_k(y)}{\sqrt{1 - y^2}} dy \\ S_k(x, \mu, \theta) &= \frac{2}{\pi} \int_{-1}^1 S(x, y, \mu, \theta) \frac{T_k(y)}{\sqrt{1 - y^2}} dy. \end{aligned}$$

*proof:*

Expanding the angular flux  $\Psi(\mathbf{x}, \mu, \theta)$  in terms of the Chebyshev polynomials in the  $y$  variable, leads to

$$\Psi(\mathbf{x}, \mu, \theta) = \sum_{i=0}^I \Psi_i(x, \mu, \theta) T_i(y). \quad (4.4)$$

Below we determine the first component, i.e.,  $\Psi_0(x, \mu, \theta)$  explicitly, whereas the other components,  $\Psi_i(x, \mu, \theta)$ ,  $i = 1, \dots, I$ , will appear as the unknowns in

$I$  one dimensional transport equations: We start to determine  $\Psi_0(x, \mu, \theta)$ , by inserting (4.4) into the boundary conditions (4.3) at  $y = \pm 1$ , to find that:

$$\Psi_0(x, \mu, \theta) = f_2(x, \mu, \phi) - \sum_{i=1}^I (-1)^i \Psi_i(x, \mu, \theta), \quad 0 < \cos \theta \leq 1, \quad (4.5)$$

$$\Psi_0(x, \mu, \theta) = - \sum_{i=1}^I \Psi_i(x, \mu, \theta), \quad -1 \leq \cos \theta < 0. \quad (4.6)$$

where  $-1 \leq x \leq 1$ ,  $-1 < \mu < 1$ , and we have used the fact that for the Chebyshev polynomials  $T_0(x) \equiv 0$ ,  $T_i(1) \equiv 1$  and  $T_i(-1) \equiv (-1)^i$ .

If we now insert  $\Psi$  from (4.4) into (4.1), multiply the resulting equation by  $\frac{T_k(y)}{\sqrt{1-y^2}}$ ,  $k = 1, \dots, I$ , and integrate over  $y$  we find that the components  $\Psi_k(x, \mu, \theta)$ ,  $k = 1, \dots, I$ , satisfy the following  $I$  one-dimensional equations:

$$\mu \frac{\partial}{\partial x} \Psi_k(x, \mu, \theta) + \sigma_t \Psi_k(x, \mu, \theta) - \int_{-1}^1 \int_0^{2\pi} \sigma_s(\mu', \phi' \rightarrow \mu, \phi) \Psi_k(x, \mu', \phi') d\theta' d\mu' + G_k(x, \mu, \theta) = 0 \quad (4.7)$$

The same procedure with the boundary condition (4.2) at  $x = 0$  and (4.4) yields

$$\Psi(0, y, \mu, \theta) = f_1(y, \mu, \phi) = \sum_{i=0}^I \Psi_i(0, \mu, \theta) T_i(y). \quad (4.8)$$

Now multiply (4.8) by  $\frac{T_k(y)}{\sqrt{1-y^2}}$ ,  $k = 1, \dots, I$ , and integrate over  $y$  we find that

$$\Psi_k(0, \mu, \theta) = \frac{2}{\pi} \int_{-1}^1 f_1(y; \mu, \theta) \frac{T_k(y)}{\sqrt{1-y^2}} dy. \quad (4.9)$$

Similarly, (note the sign of  $\mu$  below), the boundary condition at  $x = 1$  is written as

$$\sum_{i=0}^I \Psi_i(0, -\mu, \theta) T_i(y) = 0 \quad 0 < \mu \leq 1. \quad (4.10)$$

Multiplying (4.10) by  $\frac{T_k(y)}{\sqrt{1-y^2}}$ ,  $k = 1, \dots, I$  and integrating over  $y$ , we get

$$\Psi_k(0, -\mu, \theta) = 0 \quad 0 < \mu \leq 1, \quad 0 \leq \theta \leq 2\pi. \quad (4.11)$$

We can easily check that  $G_k$  in (4.7) is written as

$$G_k(x, \mu, \theta) = S_k(x, \mu, \theta) - \sqrt{1-\mu^2} \cos \theta \sum_{i=k+1}^I A_i^k \Psi_k(x, \mu, \theta) \quad (4.12)$$

where

$$A_i^k = \frac{2}{\pi} \int_{-1}^1 \frac{d}{dy} (T_i(y)) \frac{T_k(y)}{\sqrt{1-y^2}} dy \tag{4.13}$$

and

$$S_k(x, \mu, \theta) = \frac{2}{\pi} \int_{-1}^1 S(x, y, \mu, \theta) \frac{T_k(y)}{\sqrt{1-y^2}} dy. \tag{4.14}$$

Note that the solutions to the one-dimensional problems given through the equation (4.7)-(4.14) define the components  $\Psi_k(x, \mu, \theta)$ , for  $k = 1, \dots, I$ , in this decreasing order to avoid the coupling of the equations. Once this is done, the angular flux given by (4.4) is completely determined. Here we have used the convention  $\sum_{i=I+1}^I \dots = 0$ . Hence the starting  $G_I(x, \mu, \theta) \equiv S_I(x, \mu, \theta)$ . Note also that although the solution, developed in here, rely on specific boundary conditions the procedure is quite general in the sense that the expression for the first component,  $\Psi_0(x, \mu, \theta)$ , keeps the information from the boundary conditions in the  $y$  variable, while the other components are derived based on the boundary conditions in  $x$ .

### 5. ANALYSIS

Now we would like to solve the first-order linear differential equation system with isotropic scattering, i.e.,  $\sigma_s(\mu', \phi' \rightarrow \mu, \phi) \equiv \sigma_s = \text{constant}$ . Assuming isotropic scattering, the equation (4.7) is written as

$$\begin{aligned} &\mu \frac{\partial}{\partial x} \Psi_k(x, \mu, \theta) + \sigma_t \Psi_k(x, \mu, \theta) \\ &\sigma_s \int_{-1}^1 \int_0^{2\pi} \Psi_k(x, \mu', \phi') d\theta' d\mu' + G_k(x, \mu, \theta) \end{aligned} \tag{5.1}$$

for  $\mathbf{x} \in \Omega := \{(x, y): 0 \leq x \leq 1, -1 \leq y \leq 1\}$   $\mu \in [-1, 1]$  and  $\theta \in [0, 2\pi]$ .

Subject to the following boundary conditions:

For  $0 \leq \theta \leq 2\pi$ , let

$$\Psi_k(x, \mu, \theta) = \begin{cases} f_1(\mu, \phi), & \text{for } x = 0; \quad 0 < \mu \leq 1, \\ 0, & \text{for } x = 1; \quad -1 \leq \mu < 0. \end{cases}$$

The study of the problem with the anisotropic scattering is a rather involved task. See, e.g., [3] for an approach involving anisotropic scattering.

For this problem we expand the angular flux in terms of the Walsh function in the angular variable with its domain extended into the interval  $[-1, 1]$ . To this end, the Walsh function  $W_n(\mu)$  are extended in an even and odd fashion as follows [9]:

$$W_n^e(\mu) = \begin{cases} W_n(\mu), & \text{if } \mu \geq 0 \\ W_n(-\mu), & \text{if } \mu < 0 \end{cases} , \tag{5.2}$$

$$W_n^o(\mu) = \begin{cases} W_n(\mu), & \text{if } \mu \geq 0 \\ -W_n(-\mu), & \text{if } \mu < 0 \end{cases}, \quad (5.3)$$

for  $n = 0, 1, \dots, N$ . The important feature of this procedure relies the fact that a function  $f(\mu)$  defined in the interval  $[-1, 1]$  can be expanded in terms of these extended functions in the manner:

$$f(\mu) = \sum_{n=0}^{\infty} [a_n W_n^e(\mu) + b_n W_n^o(\mu)], \quad (5.4)$$

where the coefficients  $a_n$  and  $b_n$  are determined as:

$$a_n = \frac{1}{2} \int_{-1}^1 f(\mu) W_n^e(\mu) d\mu, \quad (5.5)$$

$$b_n = \frac{1}{2} \int_{-1}^1 f(\mu) W_n^o(\mu) d\mu, \quad (5.6)$$

So, in order to use the Walsh function for the solution of the problem (4.1), the angular flux is approximated by the truncated expansion:

$$\Psi_k(x, \mu, \theta) = \sum_{n=0}^N [\alpha_{nk}(x, \theta) W_n^e(\mu) + \beta_{nk}(x, \theta) W_n^o(\mu)] \quad (5.7)$$

Replacing this expansion into the linear transport equation (5.1), it turns out:

$$\sum_{n=0}^N \left[ \left\{ \mu \frac{\partial \alpha_{nk}}{\partial x}(x, \theta) + \sigma_t \alpha_{nk}(x, \theta) \right\} W_n^e(\mu) + \left\{ \mu \frac{\partial \beta_{nk}}{\partial x}(x, \theta) + \sigma_t \beta_{nk}(x, \theta) \right\} W_n^o(\mu) \right]$$

$$\begin{aligned} & \sum_{n=0}^N \sigma_s \left[ \int_{-1}^1 \int_0^{2\pi} \alpha_{nk}(x, \theta') W_n^e(\mu') d\theta' d\mu' + \int_{-1}^1 \int_0^{2\pi} \alpha_{nk}(x, \theta') W_n^o(\mu') d\theta' d\mu' \right] \\ & + G_k(x, \mu, \theta) \end{aligned} \quad (5.8)$$

Multiplying equation (5.8) by  $W_m^e$ ,  $m = 0, \dots, N$  and integrating into the interval  $[-1, 1]$ , results:

$$\begin{aligned} & \sum_{n=0}^N \left[ \frac{\partial \beta_{nk}}{\partial x}(x, \theta) \int_{-1}^1 \mu W_n^o(\mu) W_n^e(\mu) d\mu + \sigma_t \alpha_{nk}(x, \theta) \int_{-1}^1 W_n^e(\mu) W_m^e(\mu) d\mu \right] = \\ & \sum_{n=0}^N \sigma_s \left[ \int_0^{2\pi} \alpha_{nk}(x, \theta') d\theta' \int_{-1}^1 W_n^o(\mu') W_n^o(\mu') d\mu' \right] + \int_{-1}^1 G_k(x, \mu, \theta) W_n^e(\mu) d\mu \end{aligned} \quad (5.9)$$

Similarly, multiplying equation (5.8) by  $W_m^0$ ,  $m = 0, \dots, N$  and integrating yields:

$$\sum_{n=0}^N \left[ \frac{\partial \alpha_{nk}}{\partial x}(x, \theta) \int_{-1}^1 \mu W_n^o(\mu) W_n^e(\mu) d\mu + \sigma_t \beta_{nk}(x, \theta) \int_{-1}^1 W_n^0(\mu) W_m^0(\mu) d\mu \right] =$$

$$\sum_{n=0}^N \sigma_s \left[ \int_0^{2\pi} \beta_{nk}(x, \theta') d\theta' \int_{-1}^1 W_n^o(\mu') W_n^o(\mu') d\mu' \right] + \int_{-1}^1 G_k(x, \mu, \theta) W_n^0(\mu) d\mu$$

(5.10)

The integrals appearing in equations (5.9) (5.10) are known and are given [12] as

$$D_{n,m} = \frac{1}{2} \int_{-1}^1 \mu W_n^o(\mu) W_m^e(\mu) d\mu = \int_0^1 \mu W_{(n+m) \bmod 2}(\mu) d\mu$$

(5.11)

or

$$D_{n,m} = \begin{cases} 1/2, & \text{if } n = m \\ -2^{-(k+2)}, & \text{if } (n + m) \bmod 2 = 2^k, k \text{ natural} \\ 0, & \text{at another case} \end{cases}$$

(5.12)

where the notation  $(n + m) \bmod 2$  denotes the mod 2 sum of the binary digits  $n$  and  $m$  [14]

Now, following the idea of applying the Sumudu transform (cf. [8] Theorem 2.2 p. 107) to equations (5.9) and (5.10) using (5.11), we obtain an algebraic linear system

$$\sum_{n=0}^N D_{n,m} p \bar{\beta}_{n,k}(p, \theta) - \sigma_s \sum_{n=0}^N p \bar{\alpha}_{n,k}(p, \theta) + \sigma_t \bar{\alpha}_{n,k}(p, \theta) =$$

$$\int_{-1}^1 \bar{G}_k(x, \mu, \theta) W_n^e(\mu) d\mu + \sum_{n=0}^N D_{n,m} \beta_{n,k}(0, \theta)$$

(5.13)

$$\sum_{n=0}^N D_{n,m} p \bar{\alpha}_{n,k}(p, \theta) - \sigma_s \sum_{n=0}^N p \bar{\beta}_{n,k}(p, \theta) + \sigma_t \bar{\beta}_{n,k}(p, \theta) =$$

$$\int_{-1}^1 \bar{G}_k(x, \mu, \theta) W_n^o(\mu) d\mu + \sum_{n=0}^N D_{n,m} \alpha_{n,k}(0, \theta)$$

(5.14)

The equations (5.13) and (5.14) can be recast in the following matricial form:

$$\begin{bmatrix} (\sigma_t - p\sigma_s)\mathbf{I} & p\mathfrak{D} \\ p\mathfrak{D} & (\sigma_t - p\sigma_s)\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \bar{\alpha}(p, \theta) \\ \bar{\beta}(p, \theta) \end{bmatrix} = \begin{bmatrix} \mathfrak{D}\alpha(0, \theta) \\ \mathfrak{D}\beta(0, \theta) \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{G}}_1 \mathbf{I} \\ \bar{\mathbf{G}}_2 \mathbf{I} \end{bmatrix},$$

(5.15)

where  $\mathbf{I}$  is the identity of order  $N + 1$ . The matrix  $\mathfrak{D}$  has its elements defined by equation (5.11) and

$$\overline{\mathbf{G}}_1 = \int_{-1}^1 \overline{G}_k(x, \mu, \theta) W_n^e(\mu) d\mu \tag{5.16}$$

$$\overline{\mathbf{G}}_2 = \int_{-1}^1 \overline{G}_k(x, \mu, \theta) W_n^o(\mu) d\mu \tag{5.17}$$

with  $\overline{\alpha}(p, \theta)$ ,  $\overline{\beta}(p, \theta)$  and  $\overline{G}_k(x, \mu, \theta)$  denoting the Sumudu transform of the column vectors  $\alpha(x, \theta) = [\alpha_{1,k}(x, \theta) \dots \alpha_{N,k}(x, \theta)]^T$ ,  $\beta(x, \theta) = [\beta_{1,k}(x, \theta) \dots \beta_{N,k}(x, \theta)]^T$  and  $G_k(x, \mu, \theta)$ ,  $k = 1 \dots N$  respectively.

The vectors  $\overline{\alpha}(p, \theta)$  and  $\overline{\beta}(p, \theta)$  are determined solving equation (5.15) by the method of Trzaska [33] and are given as:

$$\begin{bmatrix} \alpha(x, \theta) \\ \beta(x, \theta) \end{bmatrix} = \left\{ \sum_{n=0}^N \exp(\xi_k x, \theta) \begin{bmatrix} \Delta_k^1 & \Delta_k^2 \\ \Delta_k^3 & \Delta_k^4 \end{bmatrix} \right\} \cdot \begin{bmatrix} \mathfrak{D}\beta(0, \theta) \\ \mathfrak{D}\alpha(0, \theta) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \mathbf{I} \\ \mathbf{G}_2 \mathbf{I} \end{bmatrix}, \tag{5.18}$$

where  $\xi_k$  are the roots of the characteristic polynomial of the matrix appearing on the left hand side of the equation (5.15) and  $\Delta_k^i$ ,  $i = 1, 2, 3, 4$ , are matrices resulting from the application of the *Trzaska's* method.

The unknown vectors  $\alpha(0, \theta)$  and  $\beta(0, \theta)$  are determined using the boundary conditions (4.2) and (4.3), after its expansion in terms of the extended Walsh functions. This procedure leads to the following linear system:

$$\alpha(0, \theta) + \beta(0, \theta) = \mathfrak{F} \tag{5.19}$$

$$\alpha(1, \theta) - \beta(1, \theta) = \mathbf{0} \tag{5.20}$$

where  $\mathfrak{F}$  is the vector with  $(N + 1)$  components, expressed as:

$$\mathfrak{F}_k = \int_0^1 f_1(\mu, \theta) W_k(\mu) d\mu \tag{5.21}$$

Replacing the equations (5.19) and (5.20) into equation (5.18), it turns out the following linear system for the unknown vectors  $\alpha(0, \theta)$  and  $\beta(0, \theta)$

$$\begin{bmatrix} \alpha(0, \theta) \\ \beta(0, \theta) \end{bmatrix} = \left\{ \sum_{n=0}^N \exp(0, \theta) \begin{bmatrix} \Delta_k^1 & \Delta_k^2 \\ \Delta_k^3 & \Delta_k^4 \end{bmatrix} \right\} \cdot \begin{bmatrix} \mathfrak{D}\beta(0, \theta) \\ \mathfrak{D}\alpha(0, \theta) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \mathbf{I} \\ \mathbf{G}_2 \mathbf{I} \end{bmatrix}, \tag{5.22}$$

$$\begin{bmatrix} \alpha(1, \theta) \\ \beta(1, \theta) \end{bmatrix} = \left\{ \sum_{n=0}^N \exp(\xi_k, \theta) \begin{bmatrix} \Delta_k^1 & \Delta_k^2 \\ \Delta_k^3 & \Delta_k^4 \end{bmatrix} \right\} \cdot \begin{bmatrix} \mathfrak{D}\beta(0, \theta) \\ \mathfrak{D}\alpha(0, \theta) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \mathbf{I} \\ \mathbf{G}_2 \mathbf{I} \end{bmatrix}, \tag{5.23}$$

we rewrite linear system for the unknown vectors  $\alpha(0, \theta)$  and  $\beta(0, \theta)$  as

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix} \cdot \begin{bmatrix} \alpha(0, \theta) \\ \beta(0, \theta) \end{bmatrix} + \begin{bmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{bmatrix} = \begin{bmatrix} \mathfrak{F} \\ \mathbf{0} \end{bmatrix} \quad (5.24)$$

where

$$\mathbf{A}_1 = \sum_{n=0}^N \exp(0, \theta) [\Delta_k^2 + \Delta_k^4] \mathfrak{D} \quad (5.25)$$

$$\mathbf{A}_2 = \sum_{n=0}^N \exp(0, \theta) [\Delta_k^1 + \Delta_k^3] \mathfrak{D} \quad (5.26)$$

$$\mathbf{A}_3 = \sum_{n=0}^N \exp(\xi_k, \theta) [\Delta_k^2 - \Delta_k^4] \mathfrak{D} \quad (5.27)$$

$$\mathbf{A}_4 = \sum_{n=0}^N \exp(\xi_k, \theta) [\Delta_k^1 - \Delta_k^3] \mathfrak{D} \quad (5.28)$$

$$\mathfrak{B}_1 = (\mathbf{G}_1 + \mathbf{G}_2)\mathbf{I} \quad (5.29)$$

$$\mathfrak{B}_2 = (\mathbf{G}_1 - \mathbf{G}_2)\mathbf{I} \quad (5.30)$$

So, the vectors  $\alpha(0, \theta)$  and  $\beta(0, \theta)$  are determined solving equation (5.24). Therefore the functions  $\alpha(x, \theta)$  and  $\beta(x, \theta)$  given by equation (5.18) are completely determined. Consequently, an analytical formulation is obtained for angular flux in term of the Walsh function, given by equation (5.7).

## 6. CONCLUSION

The Walsh function combined with Sumudu transform should be general enough to consider higher spatial dimensions in a way similar to that presented in this paper, although we have not investigated this idea thoroughly. We will be considering more complicated geometries in future studies, during which we will ascertain this method's usefulness for larger spatial dimensional problems. In preparation for these problems, we are currently investigating the effectiveness of spectral methods combined with Sumudu transform in solving the linear system of differential equation analytically.

An adaptation of the method for the convergence of the spectral solution within the framework of the analytical solution to study and prove convergence by using the discrete ordinates method is relatively new. The methods employing Sumudu transforms combined with Walsh function represent very interesting new ideas for studying the convergence of many numerical methods and can be extended easily to general linear transport problems. In fact only some preliminary results have been obtained. In this context we intend

to study the existence and uniqueness of its solution by using  $C_0$  semigroup approach. Our attention will be focused in this direction.

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**Received: January 10, 2007**