Common Fixed Point Theorem for
Generalized Asymptotically Nonexpansive,
Uniformly Noncommuting Mappings

P. Vijayaraju

Department of Mathematics, College of Engineering
Anna University, Chennai-600 025, Tamilnadu, India
vijay@annauniv.edu

R. Hemavathy

Department of Mathematics, Easwari Engineering College
Ramapuram, Chennai-600 089, Tamilnadu, India
hemaths@rediffmail.com

Abstract. We present common fixed point theorem for generalized nonexpansive and uniformly \( R \)-subweakly commuting mappings in normed linear spaces.

Mathematics Subject Classification: 54H25, 47H10

Keywords: Uniformly \( R \)-subweakly commuting mappings, generalized asymptotically \( f \)-nonexpansive mappings, common fixed point
1. Introduction and Preliminaries

In 1972, Goebel and Kirk [2] introduced the class of asymptotically nonexpansive mappings and the study was extended by various authors.

Recently, Cho, Sahu and Jung [1] have proved strong convergence of almost fixed points $x_n = \mu_n T^n x_n + (1 - \mu_n)q$ where $\mu_n \in (0, 1)$ to the fixed point of an asymptotically pseudocontractive mapping $T$ of a Banach space.

With the introduction of a class of R-weakly commuting mappings, Pant [5] obtained common fixed point results. Introducing a new class of noncommuting mappings namely R-subweakly commuting mappings, Shahzad [8] obtained various common fixed point and invariant approximation results for nonexpansive mappings.

More recently, Ismat Beg et al [4] extended Cho’s result to asymptotically $f$-nonexpansive mappings introducing a new class of noncommuting mappings as ”uniformly R-subweakly commuting mappings”.

The purpose of this paper is to generalize Beg’s result for generalized asymptotically $f$-nonexpansive and uniformly R-subweakly commuting mappings. Also, some extensions for generalized $(f,g)$-nonexpansive mappings were discussed.

2. Definitions and Preliminaries

Let $M$ be a nonempty subset of a normed space $X$, and let $f$, $g$ and $T$ be self mappings of $M$.

**Definition 2.1.** A mapping $T$ is said to be $(f,g)$-contraction if there exists $k \in (0,1)$ such that $\|Tx - Ty\| \leq k\|fx - gy\|$ for all $x, y \in M$. If $k = 1$, then $T$ is said to be $(f,g)$-nonexpansive.

If $f = g$ then $T$ is said to be $f$-contraction. If $k = 1$, then $T$ is said to be $f$-nonexpansive. If $f = I$ and $k = 1$, then $T$ is said to be nonexpansive.

**Definition 2.2.** A mapping $T$ is said to be asymptotically $(f,g)$-nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n \to \infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|fx - gy\|$ for all $x, y \in M$ and $n = 1, 2, 3, \ldots\infty$.

If $f = g$, then $T$ is said to be asymptotically $f$-nonexpansive.
Definition 2.3. A mapping $T$ is said to be uniformly asymptotically regular on $M$ if for each $\varepsilon > 0$, there exists $N(\varepsilon) = N$ such that $\|T^n x - T^{n+1} x\| < \varepsilon$ for all $\varepsilon \geq N$ and $x \in M$.

Definition 2.4. A mapping $f$ and $T$ are said to be $R$-weakly commuting on $M$, if there exists a real number $R > 0$ such that $\|T f x - f T x\| \leq R \|T x - f x\|$ for all $x \in M$.

Definition 2.5. A set $M$ is called $q$-starshaped with $q \in M$ if the segment $[q, x] = \{(1 - k)q + kx : 0 \leq k \leq 1\}$ is contained in $M$ for all $x \in M$.

Definition 2.6. If $M$ is $q$-starshaped with $q \in F(T)$, then $T$ and $f$ are said to be $R$-subweakly commuting on $M$ if $\|T f x - f T x\| \leq R \text{ dist}(f x, [T x, q])$ for all $x \in M$ and $R > 0$, where $\text{dist}(f x, [T x, q]) = \inf\{\|f x - y\| : y \in [T x, q]\}$.

Definition 2.7. [4] If $M$ is $q$-starshaped with $q \in F(T)$, then $T$ and $f$ are said to be uniformly $R$-subweakly commuting on $M \setminus \{q\}$ if there exists a real number $R > 0$ such that $\|T^n f x - f T^n x\| \leq R \text{ dist}(f x, [T^n x, q])$ for all $x \in M$ and $R > 0$ where $\text{dist}(f x, [T^n x, q]) = \inf\{\|f x - y_n\| : y_n \in [T^n x, q]\} = \alpha_n T^n x + (1 - \alpha_n)q : \{\alpha_n\}$ is a sequence of real numbers such that $0 < \alpha < 1$ and $\lim_{n \to \infty} \alpha_n = 1$.

It is clear from the definition, that uniformly $R$-subweakly commuting mappings on $M \setminus \{q\}$ are $R$-subweakly commuting mappings on $M \setminus \{q\}$, but $R$-subweakly commuting mappings on $M \setminus \{q\}$ need not be uniformly $R$-subweakly commuting mappings on $M \setminus \{q\}$ which can be seen from the following example.

Example 2.1. Let $X = \mathbb{R}$, with $\|x\| = |x|$, $M = [1, \infty)$ and let $T$ and $f$ be two self mappings on $M$ defined by $T x = 4x - 3$ and $f x = 2x^2 - 1$. Then $T$ and $f$ are $R$-subweakly commuting on $M \setminus \{q\}$ as $\|T f x - f T x\| \leq R \text{ dist}(f x, [T x, q])$ for all $x \in M \setminus \{q\}$ where $R = 12$ and $q = 1$ is the fixed point of $f$.

But $f$ and $T$ are not uniformly $R$-subweakly commuting on $M \setminus \{q\}$ as if we take $n = 2$, $x > 1$, then $\|T^2 f x - f T^2 x\| > R \text{ dist}(f x, [T^2 x, q])$ with $R = 12$ and $q = 1$ is the fixed point of $f$. 
Lemma 3.1. Let $M$ be a nonempty closed subset of a normed space $X$. Let $f, T : M \to M$ be self mappings, $q \in F(f)$ and $T(M \setminus \{q\}) \subset f(M \setminus \{q\})$. Suppose there exists $k \in (0, 1)$ such that
\[
\|T x - T y\| \leq k \max\{\|f x - f y\|, \|f x - T x\|, \|f y - T y\|, \frac{1}{2}(\|f x - T y\| + \|f y - T x\|)\}
\]
for all $x, y \in M$. Further, if $T$ is continuous, $\text{cl}[T(M \setminus \{q\})]$ is complete, $f$ and $T$ are $R$-weakly commuting on $M \setminus \{q\}$, then $F(f) \cap F(T)$ is singleton.

Proof. Let $x_0 \in M \setminus \{q\}$. Since $T(M \setminus \{q\}) \subset f(M \setminus \{q\})$, we can define a sequence $\{x_n\}$ in $M \setminus \{q\}$ as $f x_n = T x_{n-1}$ for each $n \geq 1$. Then
\[
\|f x_{n+1} - f x_n\| = \|T x_n - T x_{n-1}\|
\]
\[
\leq k \max\{\|f x_n - f x_{n-1}\|, \|f x_n - T x_n\|, \|f x_{n-1} - T x_{n-1}\|, \frac{1}{2}(\|f x_n - T x_{n-1}\| + \|T x_n - f x_{n-1}\|)\}
\]
\[
= k \max\{\|f x_n - f x_{n-1}\|, \|f x_n - f x_{n+1}\|, \|f x_{n-1} - f x_n\|, \frac{1}{2}(\|f x_n - f x_{n+1}\| + \|f x_{n-1} - f x_n\|)\}
\]
\[
= k \max\{\|f x_n - f x_{n-1}\|, \|f x_n - f x_{n+1}\|, \frac{1}{2}(\|f x_{n+1} - f x_{n-1}\|)\}
\]
\[
= k \max\{\|f x_n - f x_{n-1}\|, \|f x_n - f x_{n+1}\|, \frac{1}{2}(\|f x_n - f x_n\| + \|f x_n - f x_{n-1}\|)\}
\]
\[
\leq k \|f x_n - f x_{n-1}\|
\]
for all $n$. This implies that $\{f x_n\}$ is a Cauchy sequence in $M \setminus \{q\}$. So $\{T x_n\}$ is a Cauchy sequence in $M \setminus \{q\}$ and since $\text{cl}[T(M \setminus \{q\})]$ is complete, $T x_n \to y \in M$ and consequently $f x_n \to y$.

Since $T$ and $f$ are $R$-weakly commuting on $M \setminus \{q\}$,
\[
\|T f x_n - f T x_n\| \leq R \|T x_n - f x_n\|
\]
which in turn implies $fTx_n \to Ty$ as $n \to \infty$.

Now

$$
\|Tx_n - TTx_n\| \leq k \max\{\|fx_n - fTx_n\|, \|fx_n - Tx_n\|, \|fTx_n - TTx_n\|, \\
\frac{1}{2}[\|fx_n - TTx_n\| + \|fTx_n - Tx_n\|]\}
$$

Taking the limit as $n \to \infty$, we get

$$
\|y - Ty\| \leq k \max\{\|y - Ty\|, \|y - y\|, \|Ty - Ty\|, \frac{1}{2}(\|y - Ty\| + \|Ty - y\|)\}.
$$

Thus $y = Ty$.

Suppose $y \neq q$. Since $f$ and $T$ are $R$-weakly commuting on $M \setminus \{q\}$, it follows that $0 = \|fTq - Tfq\| \leq R\|Tq - fq\| = 0$, a contradiction. Since $y = Ty \in T(M \setminus \{q\})$ and $T(M \setminus \{q\}) \subset f(M) \setminus \{q\}$, there exists $z \in M \setminus \{q\}$ such that $y = fz$. Now we shall show that $fz = Tz$.

$$
\|TTx_n - Tz\| \leq k \max\{\|fTx_n - fz\|, \|fTx_n - TTx_n\|, \|fz - Tz\|, \\
\frac{1}{2}[\|fTx_n - Tz\| + \|TTx_n - fz\|]\}
$$

Now, as $n \to \infty$,

$$
\|Ty - Tz\| \leq k \max\{\|Ty - fz\|, \|Ty - Tz\|, \|fz - Tz\|, \frac{1}{2}[\|Ty - Tz\| + \|fz - Ty\|]\}
$$

$$
\|y - Tz\| \leq k \max\{\|y - Ty\|, \|y - Tz\|, \|y - Tz\|, \frac{1}{2}[\|y - Tz\| + \|y - Tz\|]\}
$$

$$
\|y - Tz\| < \|y - Tz\|.
$$

Hence $y = Tz = fz$. Since $\|fz - Tz\| \leq R\|fz - Tz\|$, we have $Tfz = fTz$.

Therefore, $y = Ty = fy$.

\[\square\]

**Theorem 3.1.** Let $M$ be a nonempty closed subset of a normed space $X$. Let $f$ and $T$ be continuous self mappings of $M$ such that $T(M \setminus \{q\}) \subset f(M) \setminus \{q\}$. Suppose $f$ is affine with respect to $q$ with $q \in F(f)$ such that $f(M) = M$ and $M$ is $q$ starshaped. If $T$ and $f$ are uniformly $R$-subweakly commuting and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n \to \infty} k_n = 1$ such that $T$ is uniformly asymptotically regular satisfying,

$$
\|T^nx - T^ny\| \leq k_n \max\{\|fx - fy\|, \text{dist}(fx, [T^nx, q]), \text{dist}(fy, [T^ny, q]), \\
\frac{1}{2}[\text{dist}(fx, [T^ny, q]) + \text{dist}(fy, [T^nx, q])]\}
$$
for all \( x, y \in M \) and \( n \in N \). Then, \( T \) and \( f \) have a common fixed point provided any one of the following conditions holds:

1. \( M \) is complete and \( \text{cl}[T(M \setminus \{q\})] \) is compact.
2. \( M \) is weakly compact and \( (f - T^n) \) is demiclosed.

Proof. For each \( n \geq 1 \), define \( T_n \) on \( M \) by \( T_n x = \mu_n T^n x + (1 - \mu_n)q \) for all \( x \in M \).

where \( \mu_n = \frac{\lambda_n}{k_n} \) and \( \{\lambda_n\} \) is a sequence of real numbers with \( 0 < \lambda_n < 1 \) such that \( \lim_{n \to \infty} \lambda_n = 1 \) and \( \{k_n\} \) is defined as above.

Now, for all \( x, y \in M \),

\[
\|T_n x - T_n y\| = \mu_n \|T^n x - T^n y\|
\]

\[
\leq \mu_n k_n \max\{\|f x - f y\|, \text{dist}(f x, [T^n x, q]), \text{dist}(f y, [T^n y, q]), \}
\]

\[
\frac{1}{2} \left( \text{dist}(f x, [T^n y, q]) + \text{dist}(f y, [T^n x, q]) \right)
\]

\[
\leq \lambda_n \max\{\|f x - f y\|, \|f x - T_n x\|, \|f y - T_n y\|, \}
\]

\[
\frac{1}{2} [\|f x - T_n y\| + \|f y - T_n x\|].
\]

Hence \( T_n \) is a generalized \( f \)-contraction for each \( n \). Also, \( T_n \) is a self mapping of \( M \) such that \( T_n(M \setminus \{q\}) \subset f(M) \setminus \{q\} \) for each \( n \). Now, from the uniformly \( R \)-subweak commutativity of \( f \) and \( T \) on \( M \setminus \{q\} \) and affineness of \( f \) with respect to \( q \), it follows that

\[
\|T_n f x - f T_n x\| = \mu_n \|T^n f x - f T^n x\|
\]

\[
\leq R \mu_n d(f x, [T^n x, q])
\]

\[
\leq R \mu_n \|f x - T_n x\|
\]

for all \( x \in M \setminus \{q\} \), which implies \( T_n \) and \( f \) are \( \mu_n R \)-weakly commuting.

Hence by Lemma 3.1, there exists \( \{x_n\} \) such that \( T_n x_n = f x_n = x_n \).

Hence \( \mu_n T^n x_n + (1 - \mu_n)q = f x_n = x_n \).

Also,

\[
\|x_n - T^n x_n\| = \|T_n x_n - T^n x_n\|
\]

\[
= \|\mu_n T^n x_n + (1 - \mu_n)q - T^n x_n\|
\]

\[
= (1 - \mu_n) \|q - T^n x_n\|
\]
Since \( T(M \setminus \{q\}) \) is bounded and as \( \mu_n \to 1 \) as \( n \to \infty \), \( \|x_n - T^n x_n\| \to 0 \) as \( n \to \infty \).

Therefore,

\[
\|x_n - T x_n\| \leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\|
\]

\[
\leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + \|T(T^n x_n) - T x_n\|
\]

\[
= \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + k_1 \max\{\|fT^n x_n - f x_n\|, \|f x_n - T x_n\|, \|fT^n x_n - fT x_n\|\}
\]

\[
\leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + k_1 \|fT^n x_n - f x_n\|
\]

\[
= \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + k_1 \|f(T^n x_n - x_n)\|.
\]

As \( f \) is continuous, affine with respect to \( q \), and \( T \) is uniformly asymptotic regular, it follows that \( \|x_n - T x_n\| \to 0 \) as \( n \to \infty \).

1. Since \( \text{cl}[T(M \setminus \{q\})] \) is compact, there exists a subsequence \( \{x_m\} \) of \( \{x_n\} \) such that \( x_m \to y \in M \) as \( m \to \infty \). As \( T \) is continuous, it follows that \( T x_m \to Ty = y \).

Moreover, \( T(M \setminus \{q\}) \subset f(M) \setminus \{q\} \) implies \( y = Ty = f z \) for some \( z \in M \).

\[
\|T x_m - T z\| \leq k_1 \max\{\|f x_m - f z\|, \|f x_m - T x_m\|, \|f z - T z\|\},
\]

\[
\frac{1}{2}[\|f x_m - T z\| + \|T x_m - f z\|]
\]

\[
\leq k_1 \|f x_m - f z\|
\]

\[
= k_1 \|x_m - y\|
\]

as \( m \to \infty \), \( T x_m \to T z \). Hence, \( y = T z = Ty = f z \).

Now, \( \|f y - Ty\| = \|f T z - T f z\| \leq R \|T z - f z\| = 0 \) which implies \( f y = Ty = y \).

2. Since \( M \) is weakly compact , there exists a subsequence \( \{x_m\} \) of \( \{x_n\} \) converging weakly to some \( y \in M \). But \( f \) being affine and continuous is weakly continuous and the weak topology is Hausdorff , it follows that \( f y = y \).

And since \( M \) is bounded, we have \( (f - T^m)x_m = (1 - m \mu_m^{-1})(x_m - q) \to 0 \) as \( m \to \infty \). Now the demiclosedness of \( (f - T^m) \) at 0 guarantees that
\[(f - T^m)y = 0. \text{ Hence } fy = T^m y = y. \text{ Now we shall show that } Ty = y.\]

\[
\|Ty - T^m y\| = \|Ty - T(T^m - 1)y\| \\
\leq k_1 \max\{\|fy - fT^m - 1 y\|, \text{ dist}(fy, [Ty, q]), \text{ dist}(fT^m - 1 y, [TT^m - 1 y, q])\} \\
\leq \frac{1}{2}\left(\text{dist}(fy, [TT^m - 1 y, q]) + \text{dist}(fT^m - 1 y, [Ty, q])\right) \\
\|Ty - y\| \leq k_1 \max\{\|y - y\|, \|fy - T_1 y\|, \|fy - T_m y\|, \\
\frac{1}{2}\|y - T_m y\| + \|y - Ty\|\} \\
\|Ty - y\| \leq \|y - Ty\| \text{ as } m \to \infty.
\]

a contradiction. Hence \(Ty = y\) which implies \(fy = Ty = y\).

The following theorem generalizes results of Cho, Sahu and Jung [1].

**Theorem 3.2.** Let \(M\) be a nonempty closed subset of a normed space \(X\). Let \(f\) and \(T\) be continuous self mappings of \(M\) such that \(T(M \setminus\{q\}) \subset f(M) \setminus\{q\}\). Suppose \(f\) is affine with respect to \(q\) with \(q \in F(f)\) such that \(f(M) = M\) and \(M\) is \(q\) starshaped. If \(T\) and \(f\) are uniformly \(R\)-subweakly commuting and \(T\) is uniformly asymptotically regular and asymptotically \(f\)-nonexpansive, then \(F(T) \cap F(f) \neq \emptyset\), provided any one of the following conditions holds:

1. \(M\) is complete and \(cl[T(M \setminus\{q\})]\) is compact.
2. \(M\) is weakly compact and \((f - T^n)\) is demiclosed.
3. \(M\) is weakly compact and \(X\) is complete satisfying Opial’s condition.

**Proof.** (1) and (2) follow from Theorem 3.1 (3) As in (2), \(fy = y\) and \(\|(f - T^m)x_m\| \to 0\) as \(m \to \infty\). If \(fy \neq T^m y\), then by Opial’s condition of \(X\) and asymptotically \(f\)-nonexpansiveness of \(T\), it follows that

\[
\lim \inf_{m \to \infty} \|fx_m - fy\| < \lim \inf_{m \to \infty} \|fx_m - T^m y\| \\
< \lim \inf_{m \to \infty} \|fx_m - T^m x_m\| + \lim \inf_{m \to \infty} \|T^m x_m - T^m y\| \\
< \lim \inf_{m \to \infty} \|T^m x_m - T^m y\| \\
\leq k_m \|fx_m - fy\|
\]

a contradiction. Hence \(fy = T^m y\). We can show that \(fy = Ty\) as in Theorem 3.1

As a direct consequence of Theorem 3.1, we have the following corollary which is Theorem 2.2 of Shahzad [9].
Corollary 3.1. Let $M$ be a nonempty closed subset of a normed space $X$. Let $f$ and $T$ be continuous self mappings of $M$ such that $T(M) \subset f(M)$. Suppose $f$ is affine with respect to $q$ with $q \in F(f)$, $M$ is $q$ starshaped and $\text{cl}[T(M)]$ is compact. If $T$ and $f$ are $R$-subweakly commuting and satisfy

$$\|Tx - Ty\| \leq \max\{\|fx - fy\|, \|fx - Tx\|, \|fy - Ty\|, \frac{1}{2}(\|fx - Ty\| + \|fy - Tx\|)\}$$

for all $x, y \in M$, then $M \cap F(T) \cap F(f) \neq \emptyset$.

Now we prove some extensions to three mappings. The following is an easy consequence of Lemma 3.1.

Lemma 3.2. Let $M$ be a nonempty closed subset of a normed space $X$. Let $f, g, T : M \to M$ be continuous self mappings, $q \in F(f)$ and $T(M \setminus \{q\}) \subset f(M) \cap g(M) \setminus \{q\}$. Suppose there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\| \leq k \max\{\|fx - gy\|, \|fx - Tx\|, \|gy - Ty\|, \frac{1}{2}(\|fx - Ty\| + \|gy - Tx\|)\}$$

(3.2)

for all $x, y \in M$. Further, $\text{cl}[T(M \setminus \{q\})]$ is complete, the pairs $\{T, f\}$ and $\{T, g\}$ are $R$-weakly commuting on $M \setminus \{q\}$, then $F(f) \cap F(g) \cap F(T)$ is singleton.

The following theorem gives an easy extension of Theorem 3.1.

Theorem 3.3. Let $M$ be a nonempty closed subset of a normed space $X$. Let $f, g$ and $T$ be continuous self mappings of $M$ such that $T(M \setminus \{q\}) \subset f(M) \cap g(M) \setminus \{q\}$. Suppose $f$ and $g$ are affine with respect to $q$ with $q \in F(f) \cap F(g)$ such that $f(M) \cap g(M) = M$ and $M$ is $q$ starshaped. If the pairs $\{T, f\}$ and $\{T, g\}$ are uniformly $R$-subweakly commuting and there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n \to \infty} k_n = 1$ such that $T$ is uniformly asymptotically regular satisfying,

$$\|T^n x - T^n y\| \leq k_n \max\{\|fx - gy\|, \text{dist}(fx, [T^n x, q]), \text{dist}(gy, [T^n y, q]), \frac{1}{2} \{ \text{dist}(fx, [T^n y, q]) + \text{dist}(gy, [T^n x, q]) \}\}$$

(3.3)

for all $x, y \in M$ and $n = 1, 2, 3, \ldots$. Then, $F(f) \cap F(g) \cap F(T) \neq \emptyset$ provided any one of the following conditions holds:
1. $M$ is complete and $\text{cl}[T(M \setminus \{q\})]$ is compact.
2. $M$ is weakly compact and $(f - T^n)$ is demiclosed.

**Proof.** As in Theorem 3.1 for each $n \geq 1$, define $T_n$ on $M$ by $T_nx = \mu_n T^nx + (1 - \mu_n)q$ for all $x \in M$.

Now, for all $x, y \in M$,

$$
\|T_nx - T_ny\| = \mu_n \|T^nx - T^ny\| \\
\leq \mu_n k_n \max\{\|fx - gy\|, \text{dist}(fx, [T^n x, q]), \text{dist}(gy, [T^n y, q]), \\
\frac{1}{2}(\text{dist}(fx, [T^n y, q]) + \text{dist}(gy, [T^n x, q]))\} \\
\leq \lambda_n \max\{\|fx - gy\|, \|fx - T_n x\|, \|gy - T_n y\|, \\
\frac{1}{2}(\|fx - T_n y\| + \|gy - T_n x\|)\}
$$

Hence each $T_n$ is a generalized $(f, g)$-contraction. Also, $T_n$ is a self mapping of $M$ such that $T_n(M \setminus \{q\}) \subset f(M) \cap g(M) \setminus \{q\}$ for each $n$. Now from the uniformly $R$-subweak commutativity of the pair $\{T, f\}$ on $M \setminus \{q\}$ and affineness of $f$ with respect to $q$, it follows that

$$
\|T_nfx - fT_nx\| = \mu_n \|T^nfx - fT^nx\| \\
\leq R\mu_n d(fx, [T^n x, q]) \\
\leq R\mu_n \|fx - T_n x\|
$$

for all $x \in M \setminus \{q\}$ which implies $\{T_n, f\}$ is $\mu_n R$-weakly commuting for each $n$. Similarly the pair $\{T_n, g\}$ is $\mu_n R$-weakly commuting for each $n$.

Hence by Lemma 3.1, there exists $\{x_n\}$ such that $T_n x_n = fx_n = gx_n = x_n \mu_n T^n x_n + (1 - \mu_n)q = fx_n = gx_n = x_n$.

Also,

$$
\|x_n - T^n x_n\| = \|T_n x_n - T^n x_n\| \\
= \|\mu_n T^n x_n + (1 - \mu_n)q - T^n x_n\| \\
= (1 - \mu_n)\|q - T^n x_n\|
$$

Since $T(M \setminus \{q\})$ is bounded and as $\mu_n \to 1$ as $n \to \infty$, $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. 

Therefore

\[
\|x_n - Tx_n\| \leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + \|T^{n+1} x_n - Tx_n\|
\]

\[
\leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + \|T(T^n x_n) - T x_n\|
\]

\[
= \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + k_1 \max\{\|f T^n x_n - g x_n\|, \|f x_n - T x_n\|, \|g T^n x_n - T T^n x_n\|, \frac{1}{2}\|f x_n - T^{n+1} x_n\| + \|g T^n x_n - T x_n\|\}
\]

\[
\leq \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + k_1 \|f T^n x_n - g x_n\|
\]

\[
= \|x_n - T^n x_n\| + \|T^n x_n - T^{n+1} x_n\| + k_1 \|f(T^n x_n - x_n)\|
\]

As \(f\) is continuous, affine with respect to \(q\), and \(T\) is uniformly asymptotic regular, it follows that \(\|x_n - Tx_n\| \to 0\) as \(n \to \infty\).

1. Since \(\text{cl}[T(M \setminus \{q\})]\) is compact, there exists a subsequence \(\{x_m\}\) of \(\{x_n\}\) such that \(x_m \to y \in M\) as \(m \to \infty\). As \(T\) is continuous, it follows that \(Tx_m \to Ty = y\).

Moreover, \(T(M \setminus \{q\}) \subset f(M) \cap g(M) \setminus \{q\}\) implies \(y = Ty = f z = g z\) for some \(z \in M\).

\[
\|Tx_m - Tz\| \leq k_1 \max\{\|f x_m - g z\|, \|f x_m - T x_m\|, \|g z - T z\|, \frac{1}{2}\|f x_m - T z\| + \|T x_m - g z\|\}
\]

\[
\leq k_1 \|f x_m - g z\|
\]

\[
\Rightarrow k_1 \|x_m - y\|
\]

as \(m \to \infty\), \(Tx_m \to T z\). Hence, \(y = T z = Ty = f z = g z\).

Now, \(\|f y - Ty\| = \|f T z - T f z\| \leq R \|T z - f z\| = 0\) which implies \(f y = Ty = y\). Similarly we can prove that \(g y = Ty = y\). Hence, \(F(f) \cap F(g) \cap F(T) \neq \emptyset\).

2. Since \(M\) is weakly compact, there exists a subsequence \(\{x_m\}\) of \(\{x_n\}\) converging weakly to some \(y \in M\). But \(f\) and \(g\) being affine and continuous are weakly continuous and the weak topology is Hausdorff, it follows that \(g y = f y = y\).

And since \(M\) is bounded, we have \((f - T^m)x_m = (1 - mu_m^{-1})(x_m - q) \to 0\) as \(m \to \infty\). Now the demiclosedness of \((f - T^m)\) at 0 guarantees that \((f - T^m)y = 0\). Hence \(g y = f y = T^m y = y\). Now we shall show that
\[ Ty = y. \]

\[
\| Ty - T^m y \| = \| Ty - T(T^{m-1}) y \| \\
\leq k_1 \max \{ \| fy - gT^{m-1} y \|, \, \dist(fy, [Ty,q]), \, \dist(gT^{m-1} y, [TT^{m-1} y, q]) \} \\
\leq k_1 \max \{ \| y - Ty \|, \| f y - T_1 y \|, \| g y - T_m y \| \} \\
\leq \| y - Ty \| \text{ as } m \to \infty.
\]

a contradiction. Hence \( Ty = y \) which implies \( gy = fy = Ty = y \).

\[ \square \]

The following theorem generalizes extends Theorem 2.3 of Hussain and Jungck [3].

**Theorem 3.4.** Let \( M \) be a nonempty closed subset of a normed space \( X \). Let \( f, g \) and \( T \) be continuous self mappings of \( M \) such that \( T(M \setminus \{q\}) \subset f(M) \cap g(M) \setminus \{q\} \). Suppose \( f \) and \( g \) are affine with respect to \( q \) with \( q \in F(f) \) such that \( f(M) \cap g(M) = M \) and \( M \) is \( q \) starshaped. If the pairs \( \{T, f\} \) and \( \{T, g\} \) are uniformly \( R \)-subweakly commuting and \( T \) is uniformly asymptotically regular and asymptotically \((f, g)\)-nonexpansive, then \( F(T) \cap F(f) \cap F(g) \neq \emptyset \), provided any one of the following conditions holds:

1. \( M \) is complete and \( \text{cl}[T(M \setminus \{q\})] \) is compact.
2. \( M \) is weakly compact and \( (f - T_m) \) is demiclosed.
3. \( M \) is weakly compact and \( X \) is complete satisfying Opial’s condition.

**Proof.** (1) and (2) follow from Theorem 3.3
(3) As in (2), \( gy = fy = y \) and \( \|(f - T^m)x_m\| \to 0 \) as \( m \to \infty \). If \( fy \neq T^m y \), then by Opial’s condition of \( X \) and asymptotically \( f \)-nonexpansiveness of \( T \),
it follows that
\[
\liminf_{m \to \infty} \|fx_m - fy\| < \liminf_{m \to \infty} \|fx_m - T^m y\|
\]
\[
< \liminf_{m \to \infty} \|fx_m - T^m x_m\| + \liminf_{m \to \infty} \|T^m x_m - T^m y\|
\]
\[
\leq k_m \|fx_m - gy\|
\]
\[
= k_m \|fx_m - fy\|
\]
\[
< \|fx_m - fy\|
\]
a contradiction. Hence \(gy = fy = T^m y\). We will show that \(fy = Ty\) as in Theorem 3.3.

As a direct consequence of Theorem 3.3, we have the following corollary which is Theorem 2.2 of Hussain and Jungck [3].

**Corollary 3.2.** Let \(M\) be a nonempty closed subset of a normed space \(X\). Let \(f, g\) and \(T\) be continuous self mappings of \(M\) such that \(T(M) \subset f(M) \cap g(M)\). Suppose \(f\) and \(g\) are affine with respect to \(q\) with \(q \in F(f) \cap F(g)\), \(M\) is \(q\) starshaped and \(cl[T(M)]\) is compact. If the pairs \(\{T,f\}\) and \(\{T,g\}\) are \(R\) -subweakly commuting and satisfy
\[
\|Tx - Ty\| \leq \max\{\|fx - gy\|, \|fx - Tx\|, \|gy - Ty\|, \frac{1}{2}(\|fx - Ty\| + \|gy - Tx\|)\}
\]
for all \(x, y \in M\), then \(M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset\).

**References**


Received: November 10, 2006