

New Extensions on Hilbert's Theorem for Double Series

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Abstract

In this paper it is shown that some new extensions of the Hilbert theorem for double series can be established by introducing a real function $\alpha(x)$ and a parameter λ . In particular, some sharp results of the classical Hilbert inequality and the Hilbert-Ingham inequality are obtained by means of Gram inequality.

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1 Introduction

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. Then

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{b}_n}{m+n} \right| \leq \pi \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |b_n|^2 \right)^{1/2} \quad (1.1a)$$

$$\left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m \bar{b}_n}{m+n+1} \right| \leq \pi \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} |b_n|^2 \right)^{1/2} \quad (1.1b)$$

and

$$\left| \sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ m \neq n}}^{\infty} \frac{a_m \bar{b}_n}{m-n} \right| \leq \pi \left(\sum_{m=1}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} |b_n|^2 \right)^{1/2} \quad (1.2a)$$

$$\left| \sum_{m=0}^{\infty} \sum_{\substack{n=0 \\ m \neq n}}^{\infty} \frac{a_m \bar{b}_n}{m-n} \right| \leq \pi \left(\sum_{m=0}^{\infty} |a_n|^2 \right)^{1/2} \left(\sum_{n=0}^{\infty} |b_n|^2 \right)^{1/2} \quad (1.2b)$$

where the constant factors π are the best possible. It is all known that the inequalities (1.1a), (1.1b), (1.2a) and (1.2b) are called the Hilbert theorems for double series. The above inequalities were combined into one similar form in some papers (such as [1]—[2] etc.), i.e.:

$$\left| \sum_{m=1-s}^{\infty} \sum_{n=1-s}^{\infty} \frac{a_m \bar{b}_n}{m+n+s} \right|^2 + \left| \sum_{m=1-s}^{\infty} \sum_{\substack{n=1-s \\ m \neq n}}^{\infty} \frac{a_m \bar{b}_n}{m-n} \right|^2 \leq \pi^2 \sum_{n=1-s}^{\infty} |a_n|^2 \sum_{n=1-s}^{\infty} |b_n|^2. \quad (1.3)$$

where $s = 0, 1$. Recently, the various extensions on (1.1a) and (1.1b) appeared in some papers (such as [3]—[4] etc.). They focalize on changing the denominators of the function of the left-hand side of (1.1a) and (1.1b). Such as the denominator $(m+n+s)$ is replaced by $(\alpha_m + \beta_n)^\mu$ in the paper [3]; and the denominator $(m+n+s)$ is replaced by $(mu(m) + nv(n))^\mu$ in the paper [4] etc. Some new results in these papers were yielded. The inequality (1.3) is obviously a significant refinement of Hilbert's inequality. However it is few that the both extension and refinement and (1.3) appear in previous papers. The main purpose of the present paper is to establish both an extension and a significant refinement on (1.3). Explicitly, let $\alpha(x); 0 \leq x < +\infty$ be a real functions, $\lim_{x \rightarrow \infty} \alpha(x) = +\infty$. $\alpha(n) = [\alpha(n)] + \frac{\lambda}{2}$. ($n \in N_0$, λ is a real number). If the denominator $(m+n+s)$ of the first term of the left-hand side of (1.3) is replaced by $\alpha(m) + \alpha(n)$, and the denominator $(m-n)$ of the second term of the left-hand side of (1.3) is replaced by $\alpha(m) - \alpha(n)$, then a new inequality established is significant in theory and applications.

For convenience, we introduce some notations and define some functions.

$$V(a, b) = \sum_{m=0}^{\infty} \sum_{\substack{n=0 \\ \alpha(m) \neq \alpha(n)}}^{\infty} \frac{a_m \bar{b}_n}{\alpha(m) - \alpha(n)}$$

$$U_k(a, b) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m \bar{b}_n}{(\alpha(m) + \alpha(n))^k}, \quad k = 1, 2.$$

$$\|x\|^2 = \sum_{n=0}^{\infty} |x_n|^2, \quad T_k(x) = \sum_{n=0}^{\infty} \frac{|x_n|}{(\alpha(n))^k}, \quad k = 1, 2.$$

$$(f, g) = \int_0^{2\pi} f(t) \overline{g(t)} dt, \quad \|Y\|^2 = \int_0^{2\pi} |Y(t)|^2 dt \quad \text{where } Y = f, g, h$$

In particular, when $b = \bar{a}$, the notations $U_k(a, a)$ and $V(a, a)$ are denoted respectively by $U_k(a)$ and $V(a)$. Throughout this paper we will frequently use these notations, and we stipulate that $\alpha(n) = [\alpha(n)] + \frac{\lambda}{2}$ ($n \in N_0$, λ is an integer or $0 < \lambda < 1$).

2 Some Lemmas

In order to prove our assertions we need the following lemmas.

Lemma 2.1. If both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent, then

- (i) $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m b_n$ is absolute convergent; (ii) $\|a\|^2$ and $\|b\|^2$ are convergent.

The proof of it has been given in the paper [2]. Hence it is omitted here.

Lemma 2.2. Let $f, g, h \in L^2[0, 2\pi]$, and h be a variable unit-vector. Then

$$|(f, g)|^2 \leq \|f\|^2 \|g\|^2 \left(1 - \left(\frac{|(f, h)|}{\|f\|} - \frac{|(g, h)|}{\|g\|} \right)^2 \right). \tag{2.1}$$

In particular, when h is orthogonal to f , we have

$$|(f, g)|^2 \leq \|f\|^2 \|g\|^2 \left(1 - \left(\frac{|(g, h)|}{\|g\|} \right)^2 \right). \tag{2.2}$$

And the equality in (2.2) holds if and only if f, g and h are linearly dependent.

Proof. Consider the determinant of Gram matrix formed by the vectors f, g and h :

$$G(f, g, h) = \begin{vmatrix} (f, f) & (f, g) & (f, h) \\ (g, f) & (g, g) & (g, h) \\ (h, f) & (h, g) & (h, h) \end{vmatrix}$$

It is known from Gram inequality that $G(f, g, h) \geq 0$. And $G(f, g, h) = 0$ if and only if f, g and h are linearly dependent. Expanding this determinant and noticing that $(h, h) = 1$, we obtain

$$\begin{aligned} (f, g)^2 &\leq \|f\|^2 \|g\|^2 - \|f\|^2 (g, h)^2 - \|g\|^2 (f, h)^2 + 2(f, g)(f, h)(g, h) \\ &\leq \|f\|^2 \|g\|^2 - (\|f\| (g, h) - \|g\| (f, h))^2. \end{aligned}$$

We point out specially that h is a variable unit-vector, it can be properly chosen in accordance with our requirement. In particular, when h selected is orthogonal to f , we have $(f, h) = 0$. It follows that Lemma 2.2 is true.

Lemma 2.3. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. Define a function by

$$r(x) = \pi U_1(x) \sin 2\lambda\pi - U_2(x) \sin^2 \lambda\pi' \quad (2.3)$$

If λ is an integer or $\frac{1}{2} \leq \lambda < 1$, then

$$r(a)r(b) \geq 0 \quad (2.4)$$

Proof. When λ is an integer, it is clear that $r(a)r(b) = 0$. So we consider only the case for $\frac{1}{2} \leq \lambda < 1$. It is easy to deduce that

$$U_1(x) = \int_0^1 \left| \sum_{m=0}^{\infty} x_m t^{\alpha(m) - \frac{1}{2}} \right|^2 dt > 0$$

$$\text{and } U_2(x) = \int_0^1 \frac{ds}{s} \int_0^s \left| \sum_{n=0}^{\infty} x_n t^{\alpha(n) - \frac{1}{2}} \right|^2 dt > 0.$$

When $\frac{1}{2} \leq \lambda < 1$, it follows from (2.3) that $r(x) < 0$ for any $x \in C$. Hence we have $r(a)r(b) > 0$.

We will frequently use the function $r(x)$, below.

3 Main Results

In order to prove our assertions we need also to introduce the following functions:

$$\begin{aligned}
 s_1(x) &= \frac{\sqrt{2} \left| T_1(x) \cos \lambda\pi - \frac{1}{2\pi} T_2(x) \sin \lambda\pi \right|}{(\pi^2 \|x\|^2 - r(x))^{1/2}} \\
 s_2(x) &= \frac{\sqrt{2} \left| T_1(x) \sin \lambda\pi - \frac{1}{\pi} T_2(x) \sin^2 \frac{\lambda\pi}{2} \right|}{(\pi^2 \|x\|^2 + r(x))^{1/2}}. \tag{3.1}
 \end{aligned}$$

Theorem 3.1. Let $r(x)$ be a function defined by (2.3), $\{a_n\}$ and $\{b_n\}$ be two nonzero sequences of complex numbers, and both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolute convergent. Then

(i) If λ is an integer, then

$$|U_1(a, b)|^2 + |V(a, b)|^2 \leq (\pi^2 \|a\|^2 \|b\|^2) (1 - R^2) \tag{3.2}$$

where

$$R^2 = \begin{cases} \frac{1}{\pi^2} \left\{ \left(\frac{T_1(a)}{\|a\|} - \frac{T_2(b)}{\pi \|b\|} \right)^2 + \left(\frac{T_1(b)}{\|b\|} - \frac{T_2(a)}{\pi \|a\|} \right)^2 \right\} & \text{if } \lambda \text{ is odd} \\ \frac{1}{\pi^2} \left(\left(\frac{T_1(a)}{\|a\|} \right)^2 + \left(\frac{T_1(b)}{\|b\|} \right)^2 \right) & \text{if } \lambda \text{ is even} \end{cases} \tag{3.3}$$

(ii) If $0 < \lambda < 1$, then

$$\begin{aligned}
 &\left| U_1(a, b) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a, b) \sin 2\lambda\pi \right|^2 + |V(a, b)|^2 \\
 &\leq \left\{ \pi^2 \|a\|^2 \|b\|^2 - \frac{1}{\pi^2} r(a) r(b) \right\} (1 - R^2) \tag{3.4}
 \end{aligned}$$

where $R^2 = \min \{ (s_1(a) - s_2(b))^2, (s_1(b) - s_2(a))^2 \}$, $s_i(x)$ ($i = 1, 2$) is defined by (3.1). In particular, when $\frac{1}{2} \leq \lambda < 1$, we have $r(a) r(b) > 0$.

Proof. Define two functions by

$$f(t) = \sum_{m=0}^{\infty} a_m \sqrt{t} \sin(\alpha(m))t, \quad g(t) = \sum_{n=0}^{\infty} b_n \sqrt{t} \cos(\alpha(n))t. \quad t \in [0, 2\pi]$$

Since the both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent, by Lemma 2.1, the double series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m b_n$ is absolute convergent. Accordingly, $f(a, t) g(b, t)$

is uniformly convergent in the interval $[0, 2\pi]$. Thereby the interchange in order of summation and integration can be made. Bellow we stipulate that the interchanges in order of summation and integration are justified. It is easy to deduce that

$$\|f\|^2 = \pi^2 \|a\|^2 - r(a), \quad \|g\|^2 = \pi^2 \|b\|^2 + r(b)$$

$$|(f, g)| = \left| \int_0^{2\pi} f(t) \overline{g(t)} dt \right| = \pi \left| V(a, b) + (U_1(a, b) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a, b) \sin 2\lambda\pi) \right|,$$

where $r(x)$ is a function defined by (2.3). By Lemma 2.2, we have

$$\begin{aligned} & \left| V(a, b) + \left(U_1(a, b) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a, b) \sin 2\lambda\pi \right) \right|^2 \\ &= \frac{1}{\pi^2} |(f, g)|^2 \leq \frac{1}{\pi^2} \|f\|^2 \|g\|^2 (1 - r) \\ &= \frac{1}{\pi^2} \{ \pi^2 \|a\|^2 - r(a) \} \{ \pi^2 \|b\|^2 + r(b) \} (1 - r) \end{aligned} \tag{3.5}$$

where $r = \left(\frac{|(f, h)|}{\|f\|} - \frac{|(g, h)|}{\|g\|} \right)^2$, h is a variable unit-vector, it can be properly chosen in accordance with our requirement.

(i) When λ is an integer, it is known from (2.3) that $r(x) = 0$.

We select $h_1 = \frac{\sqrt{2}t}{2\pi}$, it is easy to deduce that $\|h_1\| = 1$, and

$$|(f, h_1)| = \left| \sqrt{2} \left(\sum_{m=0}^{\infty} \frac{a_m}{\alpha(m)} \right) \right|. \tag{3.6}$$

Since the series $\sum_{n=0}^{\infty} a_n$ is absolute convergent, it is justified that the complex number a_n is replaced by $|a_n|$ in (3.6). Hence we have $|(f, h_1)| = \sqrt{2} T_1(a)$.

Similarly $|(g, h_1)| = \begin{cases} \frac{\sqrt{2}}{\pi} T_2(b) & \text{if } \lambda \text{ is odd} \\ 0 & \text{if } \lambda \text{ is even} \end{cases}$

We therefore obtain

$$r_1 = \left(\frac{|(f, h_1)|}{\|f\|} - \frac{|(g, h_1)|}{\|g\|} \right)^2 = \begin{cases} \frac{2}{\pi^2} \left(\frac{T_1(a)}{\|a\|} - \frac{T_2(b)}{\pi \|b\|} \right)^2 & \text{if } \lambda \text{ is odd} \\ \frac{2}{\pi^2} \left(\frac{T_1(a)}{\|a\|} \right)^2 & \text{if } \lambda \text{ is even} \end{cases} \tag{3.7}$$

Hence the inequality (3.5) can be reduced to

$$|V(a, b) + U_1(a, b)|^2 \leq \{ \pi^2 \|a\|^2 \|b\|^2 \} (1 - r_1). \tag{3.8}$$

Notice that $U_1(b, a) = \overline{U_1(a, b)}$ and $V(b, a) = -\overline{V(a, b)}$. If we still select the unit-vector $h_2 = \frac{\sqrt{2t}}{2\pi}$, then interchanging a and b in (3.8) we have

$$\left| -\overline{V(a, b)} + \overline{U_1(a, b)} \right|^2 \leq (\pi^2 \|a\|^2 \|b\|^2) (1 - r_2). \tag{3.9}$$

where r_2 is defined by

$$r_2 = \left(\frac{|(f, h_2)|}{\|f\|} - \frac{|(g, h_2)|}{\|g\|} \right)^2 = \begin{cases} \frac{2}{\pi^2} \left(\frac{T_1(b)}{\|b\|} - \frac{T_2(a)}{\pi \|a\|} \right)^2 & \text{if } \lambda \text{ is odd} \\ \frac{2}{\pi^2} \left(\frac{T_1(b)}{\|b\|} \right)^2 & \text{if } \lambda \text{ is even} \end{cases} \tag{3.10}$$

Adding (3.8) and (3.9), then the inequality (3.2) follows after simplifications

(ii) When $0 < \lambda < 1$, we firstly consider h in (3.5). We still select unit-vector $h_1 = \frac{\sqrt{2t}}{2\pi}$. It is easy to deduce that

$$|(f, h_1)| = \sqrt{2} \left| \left(\sum_{m=0}^{\infty} \frac{a_m}{u(m)} \right) \cos \lambda\pi - \frac{1}{2\pi} \left(\sum_{m=0}^{\infty} \frac{a_m}{(u(m))^2} \right) \sin \lambda\pi \right|,$$

$$|(g, h_1)| = \sqrt{2} \left| \left(\sum_{n=0}^{\infty} \frac{b_n}{u(n)} \right) \sin \lambda\pi - \frac{1}{\pi} \left(\sum_{n=0}^{\infty} \frac{b_n}{(u(n))^2} \right) \sin^2 \frac{\lambda\pi}{2} \right|.$$

Since the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent, it is justified that the complex numbers a_n and b_n are replaced respectively by $|a_n|$ and $|b_n|$ in the above relations. Let $s_1(x) = \frac{|(f, h_1)|}{\|f\|}$, $s_2(x) = \frac{|(g, h_1)|}{\|g\|}$. By using (3.1) we find $s_1(a)$, $s_2(b)$. Let $R_1^2 = (s_1(a) - s_2(b))^2$. We obtain from (3.5)

$$\begin{aligned} & \left| V(a, b) + \left(U_1(a, b) \cos 2\lambda\pi - \frac{1}{2\pi} U_2(a, b) \sin 2\lambda\pi \right) \right|^2 \\ & \leq \left\{ \pi^2 \|a\|^2 \|b\|^2 - (\|b\|^2 r(a) - \|a\|^2 r(b)) - \frac{1}{\pi^2} r(a) r(b) \right\} (1 - R_1^2). \end{aligned} \tag{3.11}$$

Notice that $U_1(b, a) = \overline{U_1(a, b)}$, $U_2(b, a) = \overline{U_2(a, b)}$ and $V(b, a) = -\overline{V(a, b)}$. If we still select the unit vector $h_2 = \frac{\sqrt{2t}}{2\pi}$, then interchanging a and b in (3.11) we have

$$\begin{aligned} & \left| -\overline{V(a, b)} + \left(\overline{U_1(a, b)} \cos 2\lambda\pi - \frac{1}{2\pi} \overline{U_2(a, b)} \sin 2\lambda\pi \right) \right|^2 \\ & \leq \left\{ \pi^2 \|a\|^2 \|b\|^2 - (\|a\|^2 r(b) - \|b\|^2 r(a)) - \frac{1}{\pi^2} r(b) r(a) \right\} (1 - R_2^2) \end{aligned} \tag{3.12}$$

where $R_2^2 = (s_1(b) - s_2(a))^2$. Let $R^2 = \min\{R_1^2, R_2^2\}$. Adding (3.11) and (3.12), the inequality (3.4) can be gotten after simplifications. In particular, when $\frac{1}{2} \leq \lambda < 1$, by Lemma 2.3, we have $r(a)r(b) \geq 0$. The proof of Theorem is completed.

Corollary 3.1. Let $\sum_{n=0}^{\infty} a_n$ be absolute convergent. Then

(i) If λ is an integer, then

$$|U_1(a)|^2 + |V(a)|^2 \leq (\pi^2 \|a\|^4) (1 - R^2) \tag{3.13}$$

where $R^2 = \begin{cases} \frac{2}{\pi^2 \|a\|^2} \left(T_1(a) - \frac{T_2(a)}{\pi}\right)^2 & \text{if } \lambda \text{ is odd} \\ \frac{2}{\pi^2 \|a\|^2} (T_1(a))^2 & \text{if } \lambda \text{ is even} \end{cases}$ (3.14)

(ii) If $0 < \lambda < 1$, then

$$\left|U_1(a) \cos 2\lambda\pi - \frac{1}{2\pi}U_2(a) \sin 2\lambda\pi\right|^2 + |V(a)|^2 \leq \left\{\pi^2 \|a\|^4 - \frac{1}{\pi^2}r^2(a)\right\} (1 - R^2)$$
 (3.15)

where $R^2 = (s_1(a) - s_2(a))^2$, $s_i(x)$ ($i = 1, 2$) is defined by (3.1).

In particular, when $\alpha(n) = n + \frac{\lambda}{2}$, according to (3.2), we obtain a refinement of (1.3) immediately.

Corollary 3.2. If $\lambda = 0, 1$, then

$$\begin{aligned} & \left| \sum_{m=1-\lambda}^{\infty} \sum_{n=1-\lambda}^{\infty} \frac{a_m \bar{b}_n}{m+n+\lambda} \right|^2 + \left| \sum_{m=1-\lambda}^{\infty} \sum_{\substack{n=1-\lambda \\ m \neq n}}^{\infty} \frac{a_m \bar{b}_n}{m-n} \right|^2 \\ & \leq \left(\pi^2 \sum_{n=1-\lambda}^{\infty} |a_n|^2 \sum_{n=1-\lambda}^{\infty} |b_n|^2 \right) (1 - R^2) \end{aligned} \tag{3.16}$$

where $R^2 = \begin{cases} \frac{1}{\pi^2} \left\{ \left(\frac{T_1(a)}{\|a\|} - \frac{T_2(b)}{\pi\|b\|}\right)^2 + \left(\frac{T_1(b)}{\|b\|} - \frac{T_2(a)}{\pi\|a\|}\right)^2 \right\} & \text{if } \lambda = 1 \\ \frac{1}{\pi^2} \left(\left(\frac{T_1(a)}{\|a\|}\right)^2 + \left(\frac{T_1(b)}{\|b\|}\right)^2 \right) & \text{if } \lambda = 0 \end{cases}$

where $T_k(x) = \sum_{n=1-\lambda}^{\infty} \frac{|x_n|}{(n + \frac{\lambda}{2})^k}$, $k = 1, 2$.

Corollary 3.3. If $\alpha(n) = n + \frac{1}{4}$, then

$$\left| \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m \bar{b}_n}{m+n+\frac{1}{2}} \right) \right|^2 + \left| \sum_{m=0}^{\infty} \sum_{n=0, m \neq n}^{\infty} \frac{a_m \bar{b}_n}{m-n} \right|^2 \leq \left\{ \pi^2 \|a\|^2 \|b\|^2 - \frac{1}{\pi^2} r(a) r(b) \right\} (1 - R^2) \tag{3.17}$$

where $r(a) r(b) > 0$, $R^2 = \min \{ (s_1(a) - s_2(b))^2, (s_1(b) - s_2(a))^2 \}$,

$$s_1(x) = \frac{\left| -\frac{1}{\sqrt{2}\pi} T_2(x) \right|}{(\pi^2 \|x\|^2 - r(x))^{1/2}}, \quad s_2(x) = \frac{\sqrt{2} \left| T_1(x) - \frac{1}{2\pi} T_2(x) \right|}{(\pi^2 \|x\|^2 + r(x))^{1/2}}$$

$$T_k(x) = \sum_{n=0}^{\infty} \frac{|x_n|}{(n + \frac{1}{4})^k}, \quad k = 1, 2.$$

Since $\lambda = \frac{1}{2}$, it is known from Lemma 2.3 that $r(a) r(b) > 0$.

If $r(a) r(b)$ and R in (3.17) are replaced by zero, the inequality (3.17) can be reduced to

$$\left| \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m \bar{b}_n}{m+n+\frac{1}{2}} \right) \right|^2 + \left| \sum_{m=0}^{\infty} \sum_{n=0, m \neq n}^{\infty} \frac{a_m \bar{b}_n}{m-n} \right|^2 < \pi^2 \|a\|^2 \|b\|^2. \tag{3.18}$$

The inequalities (3.17) and (3.18) are refinements of the Hilbert-Ingham inequality

$$\left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m \bar{b}_n}{m+n+\frac{1}{2}} \right| \leq \pi \|a\| \|b\|.$$

We have yet a new inequality according to Theorem 3.1 (ii).

Theorem 3.2. With the assumptions as Theorem 3.1, if $\lambda = \frac{1}{4}$, then

$$\left| \frac{1}{2\pi} U_2(a, b) \right|^2 + |V(a, b)|^2 \leq \left\{ \pi^2 \|a\|^2 \|b\|^2 - \frac{1}{\pi^2} r(a) r(b) \right\} (1 - R^2)$$

where $R^2 = \min \{ (s_1(a) - s_2(b))^2, (s_1(b) - s_2(a))^2 \}'$

$$s_1(x) = \frac{\left| T_1(x) - \frac{1}{2\pi} T_2(x) \right|}{(\pi^2 \|x\|^2 - r(x))^{1/2}}, \quad s_2(x) = \frac{\left| T_1(x) - \frac{1}{2\pi} T_2(x) (\sqrt{2} - 1) \right|}{(\pi^2 \|x\|^2 + r(x))^{1/2}},$$

$$T_k(x) = \sum_{n=0}^{\infty} \frac{|x_n|}{(n + \frac{1}{8})^k}, k = 1, 2$$

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