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A Note on the Regularity of Weak Solutions to the 3-D MHD Equations[1]

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Abstract

In this note we study the regularity criteria for weak solutions to the 3-D MHD equations. It is proved that if $u \in X$ which is the closure of $C([0,T]\times\mathbb{R}^3)$ in $C([0,T];L^{3,\infty})$, or $u \in L^s(0,T;\dot{M}_{p,q}(\mathbb{R}^3))$ with $\frac{2}{s}+\frac{3}{p}=1, p>3, p\geq q$, or $\nabla u \in L^s(0,T;\dot{M}_{p,q}(\mathbb{R}^3))$ with $\frac{2}{s}+\frac{3}{p}=2, p>\frac{3}{2}$ and $p\geq q$, then the solution remains smooth on $[0,T]$. Since $L^p \subset L^{p,\infty} \subset \dot{M}_{p,q}$ if $p>q$, our results improve the results in [3] and [10] when $p=q$ and hence $\dot{M}_{p,p}=L^p$. Moreover, our results like that in [3] and [10] demonstrate that the velocity field of the fluids plays a more dominant role than the magnetic field does on the regularity of solution to the magnetohydrodynamic (MHD) equations.

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1 Introduction

We consider the viscous incompressible 3-D MHD equations:

$$u_t - \Delta u + u \cdot \nabla u - B \cdot \nabla B + \nabla p + \frac{1}{2}\nabla |B|^2 = 0, \tag{1.1}$$

$$B_t - \Delta B + u \cdot \nabla B - B \cdot \nabla u = 0, \tag{1.2}$$

$$\text{div} u = \text{div} B = 0, \tag{1.3}$$

$$(u, B)|_{t=0} = (u_0, B_0). \tag{1.4}$$

where $u := (u_1, u_2, u_3)$ is the velocity field, $B := (B_1, B_2, B_3)$ is the magnetic field, p is a scalar pressure.

Duvaut and Lions [2] and Miao, Yuan and Zhang [4] constructed a class of global weak solutions. Sermange and Temam [7] proved the local existence of classical solutions and this solution is global if $(u, B) \in L^\infty(0, T; H^1(\mathbb{R}^3))$. Wu [9] proved that if the velocity and the magnetic field satisfy

$$\int_0^T \|\nabla u(t)\|_{L^2}^4 + \|\nabla b(t)\|_{L^2}^4 \text{d}t < \infty$$

or

$$\int_0^T \|u(t)\|_{L^\infty}^2 + \|b(t)\|_{L^\infty}^2 \text{d}t < \infty,$$

then the solution (u, b) remain smooth on $[0, T]$. Very recently, C. He and Z. P. Xin [3] and Y. Zhou [10] proved that if the velocity field $u \in C([0, T]; L^3)$ or $u \in L^s(0, T; L^r(\mathbb{R}^3))$ with $\frac{2}{s} + \frac{3}{r} = 1$ and $r > 3$, or $\nabla u \in L^s(0, T; L^r(\mathbb{R}^3))$ with $\frac{2}{s} + \frac{3}{r} = 2$ and $r > \frac{3}{2}$, then the solution (u, B) is smooth on $[0, T]$. It is worthy to emphasize that there are no assumptions on the magnetic field B. In other word, these results demonstrate that the magnetic field plays less dominant role than the velocity field does in the regularity theory of solutions to MHD equations. In a certain sense, these results are consistent with the numerical simulations of Politano et al. in [6].

Let $C_{0,\sigma}^\infty(\mathbb{R}^3)$ denote the set of all C^∞ real vector-valued functions $\varphi := (\varphi_1, \varphi_2, \varphi_3)$ with compact support in $\mathbb{R}^3$, such that $\text{div}\varphi = 0$. Let V be the closure of $C_{0,\sigma}^\infty(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$. And let $L^{p,q}$ denote the standard Lorentz space [8], we denote X by the closure of $C([0, T] \times \mathbb{R}^3)$ in $C([0, T]; L^{3,\infty}(\mathbb{R}))$.

Let $1 < q \le p < \infty$, we define the homogeneous Morrey space:

$$\dot{M}_{p,q} := \left\{ f \in L^q(\mathbb{R}^3) | \|f\|_{\dot{M}_{p,q}} := \sup_{R>0} \sup_{x \in \mathbb{R}^3} R^{3(\frac{1}{p} - \frac{1}{q})} \left(\int_{B(X,R)} |f(y)|^q \text{d}y \right)^{\frac{1}{q}} < +\infty \right\}.$$

Let $1 \le p' \le q' < +\infty$. We define the following homogeneous space:

$$\dot{N}_{p',q'} := \left\{ \begin{array}{l} f \in L^{p'}/f = \sum_{k\in\mathbb{N}} g_k, \quad \text{where} \quad (g_k) \subset L^{q'}_{\text{comp}}(\mathbb{R}^3) \quad \text{and} \\ \sum_{k\in\mathbb{N}} d_k^{3(\frac{1}{p'}-\frac{1}{q'})} \|g_k\|_{L^{q'}} < +\infty \quad \text{where} \quad \forall\ k, d_k = \text{diam}(\text{supp} g_k) < +\infty \end{array} \right\}.$$

Let $1 < p' \le q' < +\infty$ and p, q such that $\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1$. Then $\dot{M}_{p,q}$ is the dual space of $\dot{N}_{p',q'}$.[1]

Let $1 \le p' \le q' < 2$ and $r = \frac{3}{p}$. There exists $C > 0$ such that $\forall u \in L^2(\mathbb{R}^3)$ and $\forall\ v \in \dot{H}^r(\mathbb{R}^3)$,

$$\|uv\|_{\dot{N}_{p',q'}} \le C\|u\|_{L^2}\|v\|_{\dot{H}^r}. \tag{1.5}$$

For the proof of this inequality, see [1].

Now we are in a position to state our main result.

Theorem 1.1 Let $(u_0, B_0) \in V$. Assume that one of the following three conditions holds:

(a) $u \in L^s(0, T; \dot{M}_{p,q}(\mathbb{R}^3))$ for $\frac{2}{s} + \frac{3}{p} = 1, p > 3$ and $p \ge q \ge 1$, (1.6)

(b) $u \in X$, (1.7)

(c) $\nabla u \in L^s(0, T : \dot{M}_{p,q}(\mathbb{R}^3))$ for $\frac{2}{s} + \frac{3}{p} = 2, p > \frac{3}{2}, p \ge q$, (1.8)

then

$$(u, B) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)). \tag{1.9}$$

Remark 1.1 Since $C([0, T]; L^3) \subset X$ and $L^p \subset L^{p,\infty} \subset \dot{M}_{p,q}$, our results generalize that in [3] and [10].

Remark 1.2 We observe that if (u, b) solves (1.1)-(1.4), then so does $(u_\lambda, b_\lambda) := (\lambda u(\lambda^2 t, \lambda x), \lambda b(\lambda^2 t, \lambda x))$, and $\|u_\lambda\|_{L^s(0,\lambda^2 T;L^p)} = \|u\|_{L^s(0,T;L^p)}$ for all $\lambda > 0$ if and only if $\frac{2}{s} + \frac{3}{p} = 1$.

2 Proof of Theorem 1.1

Firstly, we assume that (1.6) holds. We differentiate (1.1) with respect to x_i, then multiply the resulting equations by $\partial_i u$ to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\partial_i u\|_{L^2}^2 + \|\nabla\partial_i u\|_{L^2}^2 &= -\int (\partial_i u \cdot \nabla)u \cdot \partial_i u dx + \int (\partial_i B \cdot \nabla)B \cdot \partial_i u dx \\ &\quad + \int (B \cdot \nabla)\partial_i B \cdot \partial_i u dx. \end{aligned} \tag{2.1}$$

Similarly,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_i B\|_{L^2}^2 + \|\nabla\partial_i B\|_{L^2}^2 &= -\int(\partial_i u\cdot\nabla)B\cdot\partial_i B\mathrm{d}x + \int(\partial_i B\cdot\nabla)u\cdot\partial_i B\mathrm{d}x \\ &\quad + \int(B\cdot\nabla)\partial_i u\cdot\partial_i B\mathrm{d}x.\end{aligned} \tag{2.2}$$

Adding (2.1) and (2.2), we obtain, by integration by parts, that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\partial_i u\|_{L^2}^2 + \|\partial_i B\|_{L^2}^2\right) + \|\nabla\partial_i u\|_{L^2}^2 + \|\nabla\partial_i B\|_{L^2}^2 \\ &\quad = -\int(\partial_i u\cdot\nabla)u\cdot\partial_i u\mathrm{d}x + \int(\partial_i B\cdot\nabla)B\cdot\partial_i u\mathrm{d}x \\ &\qquad -\int(\partial_i u\cdot\nabla)B\cdot\partial_i B\mathrm{d}x + \int(\partial_i B\cdot\nabla)u\cdot\partial_i B\mathrm{d}x =: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{2.3}$$

Each term I_i can be bounded as follows

$$\begin{aligned} I_1 &= -\int(\partial_i u\cdot\nabla)u\cdot\partial_i u\mathrm{d}x \\ &= -\sum_{j=1}^{3}\sum_{k=1}^{3}\int\partial_i u_k\cdot\partial_k u_j\cdot\partial_j u_j\mathrm{d}x = \sum_{j=1}^{3}\sum_{k=1}^{3}\int(\partial_i u_k)u_j(\partial_i\partial_k u_j)\mathrm{d}x \\ &\le \int|u||\nabla u||D^2u|\mathrm{d}x \le \|u\|_{\dot M_{p,q}}\||\nabla u||D^2 u|\|_{\dot N_{p',q'}} \\ &\le \|u\|_{\dot M_{p,q}}\|D^2u\|_{L^2}\|\nabla u\|_{\dot H^{\frac{3}{p}}} \\ &\le C\|u\|_{\dot M_{p,q}}\|D^2u\|_{L^2}^{1+\frac{3}{p}}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\end{aligned} \tag{2.4}$$

by the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{\dot H^{\frac{3}{p}}} \le C\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|D^2u\|_{L^2}^{\frac{3}{p}}. \tag{2.5}$$

Similarly,

$$\begin{aligned} I_2 &= \sum_{j=1}^{3}\int\partial_i B\cdot\nabla B_j\cdot\partial_i u_j\mathrm{d}x = -\sum_{j=1}^{3}\int(\partial_i\partial_i B\cdot\nabla B_j + \partial_i B\cdot\nabla\partial_i B_j)u_j\mathrm{d}x \\ &\le C\int|u||\nabla B||D^2B|\mathrm{d}x \le C\|u\|_{\dot M_{p,q}}\|D^2B\|_{L^2}\|\nabla B\|_{\dot H^{\frac{3}{p}}} \\ &\le C\|u\|_{\dot M_{p,q}}\|D^2B\|_{L^2}^{1+\frac{3}{p}}\|\nabla B\|_{L^2}^{1-\frac{3}{p}},\end{aligned} \tag{2.6}$$

$$\begin{aligned} I_3 &= -\int(\partial_i u\cdot\nabla)B\cdot\partial_i B\mathrm{d}x = \int u(\nabla\partial_i B\cdot\partial_i B + \nabla B\cdot\partial_i\partial_i B)\mathrm{d}x \\ &\le C\int|u||\nabla B||D^2B|\mathrm{d}x \\ &\le C\|u\|_{\dot M_{p,q}}\|D^2B\|_{L^2}^{1+\frac{3}{p}}\|\nabla B\|_{L^2}^{1-\frac{3}{p}},\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
I_4 &= \int(\partial_i B\cdot\nabla)u\cdot\partial_i B\mathrm{d}x=\sum_{j=1}^{3}\sum_{k=1}^{3}\int\partial_i B_k\cdot\partial_k u_j\cdot\partial_i B_j\mathrm{d}x\\
&= -\sum_{j=1}^{3}\sum_{k=1}^{3}\int\partial_i B_k\cdot\partial_i\partial_k B_j\cdot u_j\mathrm{d}x\le C\int|u||\nabla B||D^2B|\mathrm{d}x\\
&\le C\|u\|_{\dot{M}_{p,q}}\|D^2B\|_{L^2}^{1+\frac{3}{p}}\|\nabla B\|_{L^2}^{1-\frac{3}{p}}.
\end{aligned}
\tag{2.8}
$$

Substituting (2.4),(2.6), (2.7) and (2.8) into (2.3) and summing over i, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+\|D^2u\|_{L^2}^2+\|D^2B\|_{L^2}^2\\
&\quad\le C\|u\|_{\dot{M}_{p,q}}\|\nabla u\|_{L^2}^{1-\frac{3}{p}}\|D^2u\|_{L^2}^{1+\frac{3}{p}}\\
&\qquad+C\|u\|_{\dot{M}_{p,q}}\|\nabla B\|_{L^2}^{1-\frac{3}{p}}\|D^2B\|_{L^2}^{1+\frac{3}{p}}\\
&\quad\le\frac{1}{2}\|D^2u\|_{L^2}^2+\frac{1}{2}\|D^2B\|_{L^2}^2+C\|u\|_{\dot{M}_{p,q}}^{\frac{2p}{p-3}}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)
\end{aligned}
$$

by Young's inequality and hence

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+\|D^2u\|_{L^2}^2+\|D^2B\|_{L^2}^2\\
&\qquad\le C\|u\|_{\dot{M}_{p,q}}^{s}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)
\end{aligned}
\tag{2.9}
$$

which implies the result by Gronwall's inequality.

If $u\in X$, then we can decompose $u=v+w$ with

$$
\|v\|_{C([0,T];L^{3,\infty})}\le\varepsilon\quad\text{and}\quad\|w\|_{L^\infty((0,T)\times\mathbb{R}^3)}\le C\quad\text{for any}\quad\varepsilon>0.
$$

Then I_1 can be bounded as

$$
\begin{aligned}
I_1 &\le C\int|u||\nabla u||D^2u|\mathrm{d}x\le C\|v\|_{L^{3,\infty}}\|\nabla u\|_{L^{6,2}}\|D^2u\|_{L^2}\\
&\quad+C\|w\|_{L^\infty}\|\nabla u\|_{L^2}\|D^2u\|_{L^2}\le C\varepsilon\|D^2u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2.
\end{aligned}
$$

Here we have used the Sobolev inequality [5]:

$$
\|\nabla u\|_{L^{6,2}}\le C\|D^2u\|_{L^2}
$$

and the Hölder's inequality [8]:

$$
\|fg\|_{L^{p,q}}\le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}
$$

if $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$.

The other terms I_2, I_3 and I_4 can be treated as before, so we can obtain an inequality similar to (2.9). Then we deduce the result.

Finally, let us assume that (1.8) holds. Multiplying (1.1) by Δu, integration by parts and taking the divergence free property into account, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2 &= -\int \partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j \mathrm{d}x+\int \partial_i B_k\cdot\partial_k B_j\cdot\partial_i u_j\mathrm{d}x\\ &\quad -\int B_k\cdot\partial_i\partial_k u_j\cdot\partial_i B_j\mathrm{d}x,\end{aligned} \tag{2.10}$$

similarly,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla B\|_{L^2}^2+\|\Delta B\|_{L^2}^2 &= -\int \partial_i u_k\cdot\partial_k B_j\cdot\partial_i B_j \mathrm{d}x+\int \partial_i B_k\cdot\partial_k u_j\cdot\partial_i B_j\mathrm{d}x\\ &\quad +\int B_k\cdot\partial_k\partial_i u_j\cdot\partial_i B_j\mathrm{d}x,\end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+\|\Delta u\|_{L^2}^2+\|\Delta B\|_{L^2}^2\\ &\quad=-\int \partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j\mathrm{d}x+\int \partial_i B_k\cdot\partial_k B_j\cdot\partial_i u_j\mathrm{d}x\\ &\qquad-\int \partial_i u_k\cdot\partial_k B_j\cdot\partial_i B_j\mathrm{d}x+\int \partial_i B_k\cdot\partial_k u_j\cdot\partial_i B_j\mathrm{d}x\\ &\quad=: J_1+J_2+J_3+J_4.\end{aligned} \tag{2.12}$$

Then we estimate the above terms one by one

$$\begin{aligned}J_1 &\le \|\nabla u\|_{\dot{M}_{p,q}}\|\nabla u\|_{L^2}\|\nabla u\|_{\dot{H}^{\frac{3}{p}}}\le C\|\nabla u\|_{\dot{M}_{p,q}}\|\nabla u\|_{L^2}^{2-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}\\ &\le \varepsilon\|\Delta u\|_{L^2}^2+C\|\nabla u\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}\|\nabla u\|_{L^2}^2,\end{aligned}$$

for any $\varepsilon>0$ due to (2.5).

Similarly,

$$J_2,J_3,J_4\le\|\nabla u\|_{\dot{M}_{p,q}}\|\nabla B\|_{L^2}\|\nabla B\|_{\dot{H}^{\frac{3}{p}}}\le\varepsilon\|\Delta B\|_{L^2}^2+C\|\nabla u\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}\|\nabla B\|_{L^2}^2.$$

Inserting these estimates into (2.12) and taking ε small enough, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+\|\Delta u\|_{L^2}^2+\|\Delta B\|_{L^2}^2\\ &\quad\le C\|\nabla u\|_{\dot{M}_{p,q}}^{\frac{2p}{2p-3}}\left(\|\nabla u\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right),\end{aligned}$$

Now the Gronwall's inequality gives the result (1.9).

This completes the proof. □

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