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The Analytical Solution of a System of Nonlinear Differential Equations

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Abstract

In this paper, the analytical solution of a system of nonlinear differential equations is obtained in the reproducing kernel space $W_2^2[0,1]$. The exact solution is represented in the form of series. The $n-$term approximation $y_{j,n}(x), j = 1, 2, \cdots, p$ are proved to converge to exact solution $y_j(x), j = 1, 2, \cdots, p$. More importantly, the method's implementation requires no additional conditions. Some examples are presented to demonstrate the reliability and efficiency of the algorithm developed.

Keywords: Analytical solution, nonlinear system of differential equations, reproducing kernel

1 Introduction

Systems of differential equations which are often encounters in applications. Most realistic systems of ordinary differential equations do not have analytical solutions so that the numerical technique must be used. They can be readily solved by many methods, such as the simple Taylor series method and fourth-order Runge-Kutta method [1,2],the Tau Method[3,4,5] and the Adomian's decomposition method [7,8]. Over the past few years, many new alternatives to the use of traditional methods for the numerical solution of systems of differential equations have been proposed. In ref.[6], the author present the operational approach to the Tau Method for the numerical solution of mixed-order systems of linear ordinary differential equations with polynomial or rational

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polynomial coefficients, together with initial or boundary conditions. Dogan Kaya deals with the implementation of the Adomian's decomposition method in chemical applications[9]. In ref. [10], the Adomian's decomposition method is applied to initial problems for systems of ordinary differential equations in both linear and nonlinear cases.

In this paper, we will consider general problems mainly focus on systems of nonlinear differential equations. Since every ordinary differential equation of order n can be written as a systems consisting of n first-order ordinary differential equations, we restrict our study to a system of first-order differential equations . We consider the following systems of nonlinear ordinary differential equations:

$$Ly_j \triangleq \frac{dy_j}{dx} = f_j(x, y_1(x), y_2(x), \cdots, y_p(x)), y_j(0) = 0, j = 1, 2, \cdots, p, \qquad (1.1)$$

where, $y_1(x), y_2(x), \cdots, y_p(x)$ is the solution of (1.1), and $y_j(x) \in W_2^2[0,1], j = 1, 2, \cdots, p$. In this paper, the exact solution of (1.1) is obtained in the reproducing kernel space $W_2^2[0,1]$. The analytical solution is represented in the form of series. The $n-$term approximation $y_{j,n}(x), j = 1, 2, \cdots, p$ are proved to converge to the exact solution $y_j(x), j = 1, 2, \cdots, p$.

2 Preliminary

In this section, the reproducing kernel space is defined to solve (1.1). Space $W_2^2[0,1]$ is defined as
$W_2^2[0,1] = \{u | u'$ is absolutely continuous function and $u'' \in L^2[0,1], u(0) = 0\}$. The inner product $(\cdot,\cdot)$ and the the norm $\|\cdot\|_{W_2^2[0,1]}$ are taken to be

$$(u(y), v(y)) \overset{\text{def}}{=} \int_0^1 (4uv + 5u'v' + u''v'')dy, u, v \in W_2^2[0,1] \qquad (2.1)$$

$$\|u\|_{W_2^2} \overset{\text{def}}{=} \sqrt{(u,u)}$$

respectively.

Space $W_2^1[0,1]$ is defined by $W_2^1[0,1] = \{u | u$ is absolutely continuous function and $u' \in L^2[0,1]\}$ equipped with the inner product

$$\langle u, v\rangle \overset{\text{def}}{=} \int_0^1 uvdx + \int_0^1 u'v'dx, \qquad u, v \in W_2^1[0,1]$$

and norm

$$\|u\|_{W_2^1} \overset{\text{def}}{=} \sqrt{\langle u,u\rangle}$$

respectively.

In reference [11] the authors proved that $W_2^2[0,1], W_2^1[0,1]$ are reproducing kernel spaces with the kernels

$$R_x(y) = \begin{cases} R_1(x,y), y \le x; \\ R_2(x,y), y > x; \end{cases} \tag{2.2}$$

where

$$\begin{aligned} R_1(x,y) &= \frac{e^{-2x-2y}(9e^4+7e^6-7e^{4x}-8e^{3+x}+8e^{3+3x}-9e^{2+4x})}{12(-7-9e^2+9e^4+7e^6)} \\ &+\frac{e^{-2x+2y}(-9e^4-7e^6+7e^{4x}+8e^{3+x}-8e^{3+3x}+9e^{2+4x})}{12(-7-9e^2+9e^4+7e^6)} \\ &+\frac{e^{-2x+y}(4e^3-7e^{3x}-9e^{2+x}+7e^{6+x}+9e^{4+3x}-4e^{3+4x})}{6(-7-9e^2+9e^4+7e^6)} \\ &+\frac{e^{-2x-y}(-4e^3+7e^{3x}+9e^{2+x}-7e^{6+x}-9e^{4+3x}+4e^{3+4x})}{6(-7-9e^2+9e^4+7e^6)} \end{aligned}$$

$$\begin{aligned} R_2(x,y) &= \frac{e^{-2x+2y}(-1+e^{2x})(7+9e^2+7e^{2x}-8e^{3+x}+9e^{2+2x})}{12(-7-9e^2+9e^4+7e^6)} \\ &+\frac{e^{3-2x-2y}(-1+e^{2x})(9e+7e^3-8e^x+9e^{1+2x}+7e^{3+2x})}{12(-7-9e^2+9e^4+7e^6)} \\ &+\frac{e^{2-2x-y}(-4e+9e^x-9e^{3x}-7e^{4+x}+7e^{4+3x}+4e^{1+4x})}{6(-7-9e^2+9e^4+7e^6)} \\ &+\frac{e^{-2x+y}(4e^3+7e^x-7e^{3x}-9e^{4+x}+9e^{4+3x}-4e^{3+4x})}{6(-7-9e^2+9e^4+7e^6)} \end{aligned}$$

and

$$\bar{R}_x(y) = \frac{1}{2\sinh(1)}[\cosh(x+y-1)+\cosh(|x-y|-1)].$$

Then, for any $u \in W_2^2[0,1], v \in W_2^1[0,1]$ and a fixed x, it follows that

$$u(x) = (u(\xi), R_x(\xi)), v(x) = \langle v(\xi), \overline{R}_x(\xi)\rangle.$$

3 The method

In this section, the analytical solution of (1.1) is given in the reproducing kernel space $W_2^2[0,1]$. It is clear that $L : W_2^2[0,1] \to W_2^1[0,1]$ is a bounded linear operator. Let $\varphi_i(x) = \overline{R}_{x_i}(x), \overline{R}_x(y)$ is the reproducing kernel of $W_2^1[0,1]$, and $\psi_i(x) = L^*\varphi_i(x)$, L^* is the conjugate operator of L.

Theorem 3.1. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$*, then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete system of* $W_2^2[0,1]$*.*

Proof. For $\forall u(x) \in W_2^2[0,1]$, let $(\psi_i(x), u(x)) = 0$, It follows

$$\langle \varphi_i(x), Lu(x)\rangle = Lu(x_i) = 0, 2, i = 1, 2, \cdots$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on [0,1], hence, $Lu(x) = 0$. It follows that $u(x) = 0$ from the existence of L^-. ∎

Practise Gram-Schmidt orthonomalization for $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\tilde{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x). \tag{3.1}$$

Then $\{\tilde{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of $W_2^2[0,1]$.

Theorem 3.2. *Let $\{x_i\}_{i=1}^{\infty}$ be dense on $[0,1]$, if the solution of (1.1) is unique, then the solution satisfies the form*

$$y_j(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f_j(x_k, y_1(x_k), y_2(x_k)\cdots, y_p(x_k))\tilde{\psi}_i(x), j = 1, 2, \cdots, p. \tag{3.2}$$

Proof. Note that $\langle u(x), \varphi_i(x)\rangle = u(x_i)$ and $\{\tilde{\psi}_i(x)\}_{i=1}^{\infty}$ is an orthonormal basis of $W_2^2[0,1]$, hence

$$\begin{aligned} y_j(x) &= \textstyle\sum_{i=1}^{\infty}(y_j(x), \tilde{\psi}_i(x))\tilde{\psi}_i(x) \\ &= \textstyle\sum_{i=1}^{\infty}(y_j(x), \sum_{k=1}^{i}\beta_{ik}\psi_k(x))\tilde{\psi}_i(x) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}(y_j(x), \psi_k(x))\tilde{\psi}_i(x) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Ly_j(x), \varphi_k(x)\rangle\tilde{\psi}_i(x) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f_j(x_k, y_1(x_k), y_2(x_k), \cdots, y_p(x_k))\tilde{\psi}_i(x), 1 \le j \le p. \end{aligned} \tag{3.3}$$

□

4 Implementation

In this section, a new method of solving (1.1) is presented. (3.2) can be denoted by

$$y_j(x) = \sum_{i=1}^{\infty} A_i^{(j)}\tilde{\psi}_i(x), j = 1, 2, \cdots, p, \tag{4.1}$$

where

$$A_i^{(j)} = \sum_{k=1}^{i} \beta_{ik} f_j(x_k, y_1(x_k), y_2(x_k), \cdots, y_p(x_k)).$$

Let $x_1 = 0$, it follows that $y_1(x_1), y_2(x_1), \cdots, y_p(x_1)$ are known from the initial conditions. So $f_j(x_1, y_1(x_1), y_2(x_1), \cdots, y_p(x_1))(j = 1, 2, \cdots p)$ are known. Considering the numerical computation, we put $y_{1,0}(x_1) = y_1(x_1), y_{2,0}(x_1) =$

$y_2(x_1), \cdots, y_{p,0}(x_1) = y_p(x_1)$ and the $n-$term approximation to $y_j(x)$, $j = 1, 2, \cdots, p$ by

$$y_{j,n}(x) = \sum_{i=1}^{n} B_i^{(j)} \tilde{\psi}_i(x), j = 1, 2 \cdots, p \tag{4.2}$$

where

$$\begin{array}{lcl} B_1^{(j)} & = & \beta_{11} f_j(x_1, y_{1,0}(x_1), y_{2,0}(x_1), \cdots, y_{p,0}(x_1)), \\ y_{j,1}(x) & = & B_1^{(j)} \tilde{\psi}_1(x), \\ B_2^{(j)} & = & \sum_{k=1}^{2} \beta_{2k} f_j(x_k, y_{1,k-1}(x_k), y_{2,k-1}(x_k), \cdots, y_{p,k-1}(x_k)), \\ y_{j,2}(x) & = & \sum_{i=1}^{2} B_i^{(j)} \tilde{\psi}_i(x), \\ & & \cdots\cdots \\ B_n^{(j)} & = & \sum_{k=1}^{n} \beta_{nk} f_j(x_k, y_{1,k-1}(x_k), y_{2,k-1}(x_k), \cdots, y_{p,k-1}(x_k)), \end{array} \tag{4.3}$$

$j = 1, 2, \cdots, p$.

Lemma 4.1. *If $y_j(x) \in W_2^2[0,1], j = 1, 2, \cdots, p$, then there exists a constant $C > 0$, such that $|y_j(x)| \leq C\|y_j(x)\|_{W_2^2}$, $|y_j'(x)| \leq C\|y_j(x)\|_{W_2^2}, j = 1, 2, \cdots, p$.*

Proof. For any $y_j(x) \in W_2^2[0,1]$,

$$|y_j(x)| = |(y_j(\xi), R_x(\xi))| \leq \|y_j(x)\|_{W_2^2} \|R_x(\xi)\|_{W_2^2}$$

then there exists a $C_j > 0$, such that

$$|y_j(x)| \leq C_j \|y_j(x)\|_{W_2^2}, j = 1, 2, \cdots, p,$$

Note that

$$|y_j'(x)| = (y_j(\xi), \frac{d}{dx} R_x(\xi)) \leq \|y_j(x)\|_{W_2^2} \|\frac{d}{dx} R_x(\xi)\|_{W_2^2}, j = 1, 2, \cdots, p,$$

Hence there exists a $C_j' > 0$, such that

$$|y_j'(x)| \leq C_j' \|y_j\|_{W_2^2}.$$

Put $C = max\{C_j, C_j', j = 1, 2, \cdots, p\}$, then the proof is complete. ☐

Lemma 4.2. *If $y_{j,n}(x) \stackrel{\|\cdot\|_{W_2^2}}{\longrightarrow} \overline{y}_j(x) (n \to \infty)$, and $\|y_{j,n}(x)\|, j = 1, 2, \cdots, p$ are bounded, then $f_j(x_n, y_{1,n-1}(x_n), y_{2,n-1}(x_n), \cdots, y_{j,n-1}(x_n))$* converges to *$f_j(x, \overline{y}_1(x), \overline{y}_2(x), \cdots, \overline{y}_p(x)) (n \to \infty), j = 1, 2, \cdots, p$.*

Proof. From the given condition $y_{j,n}(x) \overset{\|\cdot\|_{W_2^2}}{\longrightarrow} \overline{y}_j(x)(n \to \infty), j = 1, 2, \cdots, p$ and Lemma 4.1, it follows that, for $x \in [0, 1]$

$$|y_{j,n-1}(x) - \overline{y}_j(x)| \to 0(n \to \infty), |y'_{j,n-1}(\xi)| \le C\|y_{j,n-1}(x)\|_{W_2^2},$$

Observing that

$$\begin{aligned} |y_{j,n-1}(x_n) - \overline{y}_j(x)| &= |y_{j,n-1}(x_n) - y_{j,n-1}(x) + y_{j,n-1}(x) - \overline{y}_j(x)|, \\ &= \le |y'_{j,n-1}(\xi)||x_n - x| + |y_{j,n-1}(x) - \overline{y}_j(x)|, \end{aligned}$$

by the boundedness of $\|y_{j,n}\|_{W_2^2}$, we get

$$|y_{j,n-1}(x_n) - \overline{y}_j(x)| \to 0, n \to \infty,$$

The continuation of $f_j(x, y_1(x), y_2(x), \cdots, y_p(x))$ implies that $f_j(x_n, y_{1,n-1}(x_n), y_{2,n-1}(x_n), \cdots, y_{j,n-1}(x_n)) \to f_j(x, \overline{y}_1(x), \overline{y}_2(x), \cdots, \overline{y}_p(x))$ $(n \to \infty)$, $j = 1, 2, \cdots, p$. □

Theorem 4.1. *Assume $\|y_{j,n}(x)\|_{W_2^2}$ are bounded in (4.2), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, then $n-$term approximate solution $y_{j,n}(x)$ converges to the exact solution $y_j(x), j = 1, 2, \cdots, p$ of (1.1), and the exact solution is expressed as*

$$y_j(x) = \sum_{i=1}^{\infty} B_i^{(j)} \tilde{\psi}_i(x), j = 1, 2, \cdots, p, \tag{4.4}$$

where $B_i^{(j)}$ is given by (4.3).

Proof. First we will prove the convergence of $y_{j,n}(x), j = 1, 2, \cdots, p$.

$$(y_{j,n}(x), y_{j,n}(x)) = (\sum_{i=1}^{n} B_i^{(j)} \tilde{\psi}_i(x), \sum_{i=1}^{n} B_i^{(j)} \tilde{\psi}_i(x)) = \sum_{i=1}^{n} (B_i^{(j)})^2. \tag{4.5}$$

Namely $\|y_{j,n}\|_{W_2^2}^2 = \sum_{i=1}^{n} (B_i^{(j)})^2$, it implies that $\|y_{j,n}(x)\|, j = 1, 2, \cdots, p$ are monotonically increasing functions. Due to the conditions that $\|y_{j,n}(x)\|_{W_2^2}$ is bounded, hence $\|y_{j,n}(x)\|_{W_2^2}$ is convergent and there exists a constant c, such that

$$\sum_{i=1}^{\infty} (B_i^{(j)})^2 = c, j = 1, 2, \cdots, p.$$

Hence

$$B_i^{(j)} \in l^2, j = 1, 2, \cdots, p.$$

If $m > n$, then

$$\|y_{j,m}(x) - y_{j,n}(x)\|_{W_2^2}^2 = \|\sum_{i=n+1}^{m} B_i^{(j)} \tilde{\psi}_i(x)\|_{W_2^2}^2 = \sum_{i=n+1}^{m} (B_i^{(j)})^2 \to 0, m, n \to \infty,$$

Considering the completeness of $W_2^2[0,1]$, we get

$$y_{j,n}(x) \xrightarrow{\|\cdot\|} \overline{y}_j(x), n \to \infty.$$

Second, we will prove that $\overline{y}_j(x), j = 1, 2, \cdots, p$ is the solution of Eq.(1.1)

On taking limits in (4.2)

$$\overline{y}_j(x) = \sum_{i=1}^{\infty} B_i^{(j)} \tilde{\psi}_i(x), j = 1, 2, \cdots, p \tag{4.6}$$

Since
$(L\overline{y}_j)(x_n) = \sum_{i=1}^{\infty} B_i^{(j)} \langle L\tilde{\psi}_i, \varphi_n \rangle = \sum_{i=1}^{\infty} B_i^{(j)} (\tilde{\psi}_i, L^*\varphi_n) = \sum_{i=1}^{\infty} B_i^{(j)} (\tilde{\psi}_i, \psi_n), j = 1, 2, \cdots, p$, it follows that

$$\sum_{l=1}^{n} \beta_{nl}(L\overline{y}_j)(x_l) = \sum_{i=1}^{\infty} B_i^{(j)} (\tilde{\psi}_i, \sum_{l=1}^{n} \beta_{nl}\psi_l) = \sum_{i=1}^{\infty} B_i^{(j)} (\tilde{\psi}_i, \tilde{\psi}_n) = B_n^{(j)}, j = 1, 2, \cdots, p.$$

If $n = 1$, then

$$(L\overline{y}_j)(x_1) = f_j(x_1, y_{1,0}(x_1), y_{2,0}(x_1), \cdots, y_{p,0}(x_1)), j = 1, 2, \cdots, p.$$

If $n = 2$, then
$\beta_{21}(L\overline{y}_j)(x_1) + \beta_{22}(L\overline{y}_j)(x_2) = \beta_{21} f_j(x_1, y_{1,0}(x_1), y_{2,0}(x_1), \cdots, y_{p,0}(x_1))$
$+ \beta_{22} f_j(x_2, y_{1,1}(x_2), y_{2,1}(x_2), \cdots, y_{p,1}(x_2)) j = 1, 2, \cdots, p.$
It is clear that

$$(L\overline{y}_j)(x_2) = f_j(x_2, y_{1,1}(x_2), y_{2,1}(x_2), \cdots, y_{p,1}(x_2)), j = 1, 2, \cdots, p.$$

Moreover, it is easy to see by induction that

$$(L\overline{y}_j)(x_l) = f_j(x_l, y_{1,l-1}(x_l), y_{2,l-1}(x_l), \cdots, y_{p,l-1}(x_l)), l = 1, 2 \cdots, j = 1, 2, \cdots, p. \tag{4.7}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, for $\forall z \in [0,1]$, there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_l} \to z, l \to \infty$.
Hence, let $l \to \infty$ in (4.7), by the convergence of $y_{j,n}$ and Lemma 4.2, we have

$$(L\overline{y}_j)(z) = f_j(z, \overline{y}_1(z), \overline{y}_2(z), \cdots, \overline{y}_p(z)), j = 1, 2, \cdots, p. \tag{4.8}$$

From (4.8), it follows that $\overline{y}_j, j = 1, 2, \cdots, p$ satisfies (1.1). Since $\tilde{\psi}_i(x) \in W_2^2[0,1]$, clearly, $\overline{y}_j, j = 1, 2, \cdots, p$ satisfies the initial conditions of (1.1). That is, $\overline{y}_j, j = 1, 2, \cdots, p$ is the solution of (1.1). The application of uniqueness of solution of (1.1) then yields that

$$y_j(x) = \sum_{i=1}^{\infty} B_i^{(j)} \tilde{\psi}_i(x), j = 1, 2, \cdots, p. \tag{4.9}$$

□

5 Example

In this section, some numerical examples will be tested by using the method discussed above. For comparison reasons, the problems have known solutions . All experiments were performed in MATHEMATICA 5.0.

Example 1

Consider the system

$$\begin{cases} \frac{dy_1}{dx} = -y_1 + g_1(x) \\ \frac{dy_2}{dx} = y_1 - y_2^2 + g_2(x) \\ \frac{dy_3}{dx} = y_2^2 + g_3(x) \end{cases}$$

where

$$\begin{aligned} g_1(x) &= -1 + x + x^2, \\ g_2(x) &= e^{-2x}(-e^2 x(-1+x)x + (-1+x)^2 x^2 - e^x(1 - 3x + x^2), \\ g_3(x) &= -e^{-2x}(-1+x)^2 x^2 + (-1+x)cosx + sinx, \end{aligned}$$

$y_i(x) \in W_2^2[0,1], 0 \le x \le 1, i = 1,2,3$ subject to boundary conditions $y_i(0) = 0$ which has the exact solution given by $y_1(x) = x(x-1), y_2(x) = x(x-1)e^{-x}, y_3(x) = (x-1)sinx$. The exact and approximate solutions and the absolute error are displayed in Table 1 Table 2 and Table 3 with $N = 100$.

Table 1:

Node	True solution $y_1(x)$	Approximate solution $y_{1,100}(x)$	Absolute error
0.08	-0.0736	-0.0735023	9.76695E-5
0.16	-0.1344	-0.134308	9.2227E-5
0.24	-0.1824	-0.182314	8.57351E-5
0.32	-0.2176	-0.217522	7.83669E-5
0.40	-0.24	-0.23993	7.02808E-5
0.48	-0.2496	-0.249538	6.1622E-5
0.56	-0.2464	-0.246347	5.25232E-5
0.64	-0.2304	-0.230357	4.31057E-5
0.72	-0.2016	-0.201567	3.34802E-5
0.80	-0.16	-0.159976	2.37477E-5
0.88	-0.1056	-0.105586	1.40004E-5
0.96	-0.0384	-0.0383957	4.32225E-6

Example 2

Consider the system

$$\begin{cases} \frac{dy_1}{dx} = -y_1 + y_2 y_3 + g_1(x) \\ \frac{dy_2}{dx} = y_1 - y_2 y_3 - y_2^2 + g_2(x) \\ \frac{dy_3}{dx} = y_2^2 + g_3(x) \end{cases}$$

Table 2:

Node	True solution $y_2(x)$	Approximate solution $y_{2,100}(x)$	Absolute error
0.08	-0.0679414	-0.0677676	1.73724E-4
0.16	-0.114528	-0.114369	1.59168E-4
0.24	-0.143481	-0.143335	1.45558E-4
0.32	-0.15801	-0.157878	1.32289E-4
0.40	-0.160877	-0.160758	1.18947E-4
0.48	-0.154448	-0.154343	1.05254E-4
0.56	-0.140746	-0.140655	9.10305E-5
0.64	-0.121488	-0.121412	7.61638E-5
0.72	-0.0981293	-0.0980687	6.05938E-5
0.80	-0.0718926	-0.0718483	4.42973E-5
0.88	-0.0438011	-0.0437738	2.72798E-5
0.96	-0.0147031	-0.0146935	9.56886E-6

Table 3:

Node	True solution $y_3(x)$	Approximate solution $y_{3,100}(x)$	Absolute error
0.08	-0.0735215	-0.0734251	9.64426E-5
0.16	-0.133827	-0.133736	9.14412E-5
0.24	-0.180654	-0.180569	8.50779E-5
0.32	-0.213905	-0.213827	7.78107E-5
0.40	-0.233651	-0.233581	6.99262E-5
0.48	-0.240125	-0.240064	6.15973E-5
0.56	-0.233722	-0.233669	5.29219E-5
0.64	-0.21499	-0.214946	4.39486E-5
0.72	-0.165341	-0.165311	2.99632E-5
0.80	-0.143471	-0.143446	2.51599E-5
0.88	-0.0924887	-0.0924733	1.53314E-5
0.96	-0.0327677	-0.0327625	5.19213E-6

where

$$\begin{aligned} g_1(x) &= -1+x+x^2-e^{-x}(-1+x)^2xsinx, \\ g_2(x) &= e^{-2x}(-e^2x(-1+x)x+(-1+x)^2x^2-e^x(1-3x+x^2) \\ &\quad +e^x(-1+x)^2xsinx, \\ g_3(x) &= -e^{-2x}(-1+x)^2x^2+(-1+x)cosx+sinx, \end{aligned}$$

$y_i(x) \in W_2^2[0,1], 0 \leq x \leq 1, i = 1,2,3$ subject to boundary conditions $y_i(0) = 0$ which has the exact solution given by $y_1(x) = x(x-1), y_2(x) = x(x-1)e^{-x}, y_3(x) = (x-1)sinx$. The exact and approximate solutions and the absolute error are displayed in Table 4 Table 5 and Table 6 with $N = 100$.

Table 4:

Node	True solution $y_1(x)$	Approximate solution $y_{1,100}(x)$	Absolute error
0.08	-0.0736	-0.073501	9.89773E-5
0.16	-0.1344	-0.134307	9.29695E-5
0.24	-0.1824	-0.182315	8.52299E-5
0.32	-0.2176	-0.217524	7.63734E-5
0.40	-0.24	-0.239933	6.6882E-5
0.48	-0.2496	-0.249543	5.71277E-5
0.56	-0.2464	-0.246353	4.73894E-5
0.64	-0.2304	-0.230362	3.78659E-5
0.72	-0.2016	-0.201571	2.86835E-5
0.80	-0.16	-0.15998	1.99019E-5
0.88	-0.1056	-0.105588	1.15182E-5
0.96	-0.0384	-0.0383965	3.47006E-6

Table 5:

Node	True solution $y_2(x)$	Approximate solution $y_{2,100}(x)$	Absolute error
0.08	-0.0679414	-0.067769	1.7235E-4
0.16	-0.114528	-0.11437	1.58291E-4
0.24	-0.143481	-0.143335	1.45853E-4
0.32	-0.15801	-0.157876	1.34019E-4
0.40	-0.160877	-0.160755	1.2207E-4
0.48	-0.154448	-0.154339	1.09504E-4
0.56	-0.140746	-0.14065	9.59847E-5
0.64	-0.121488	-0.121407	8.13042E-5
0.72	-0.0981293	-0.0980639	6.53646E-5
0.80	-0.0718926	-0.0718445	4.81657E-5
0.88	-0.0438011	-0.0437713	2.97982E-5
0.96	-0.0147031	-0.0146926	1.04382E-5

Table 6:

Node	True solution $y_3(x)$	Approximate solution $y_{3,100}(x)$	Absolute error
0.08	-0.0735215	-0.073425	9.65043E-5
0.16	-0.133827	-0.133736	9.15754E-5
0.24	-0.180654	-0.180569	8.52884E-5
0.32	-0.213905	-0.213827	7.80743E-5
0.40	-0.233651	-0.233581	7.02021E-5
0.48	-0.240125	-0.240063	6.18419E-5
0.56	-0.233722	-0.233669	5.31015E-5
0.64	-0.21499	-0.214946	4.4048E-5
0.72	-0.184628	-0.184593	3.47209E-5
0.80	-0.143471	-0.143446	2.51372E-5
0.88	-0.0924887	-0.0924734	1.52952E-5
0.96	-0.0327677	-0.0327625	5.17496E-6

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