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Higher Order Gabitov-Turitsyn Equation for Dispersion-Managed Solitons in Multiple Channels

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Abstract

The higher order Gabitov-Turstsyn equation is derived in the context of dense wavelength-division-multiplexed soliton systems. The higher order multiple scale perturbation analysis is exploited to carry out this derivation. This equation describes the propagation of solitons in such systems with higher order accuracy.

Mathematics Subject Classification: 35Q51; 35Q55; 37K10; 78A60

1 INTRODUCTION

The propagation of solitons through optical fibers has been a major area of research given its potential applicability in all optical communication systems. The field of telecommunications has undergone a substantial evolution in the last couple of decades due to the impressive progress in the development of optical fibers, optical amplifiers as well as transmitters and receivers. In a modern optical communication system, the transmission link is composed of optical fibers and amplifiers that replace the electrical regenerators. But the amplifiers introduce some noise and signal distortion that limit the system capacity. Presently the optical systems that show the best characterestics in terms of simplicity, cost and robustness against the degrading effects of a link are those based on intensity modulation with direct detection (IM-DD). Conventional IM-DD systems are based on non-return-to-zero (NRZ) format, but for transmission at higher data rate the return-to-zero (RZ) format is preferred. When the data rate is quite high, soliton transmission can be used. It allows the exploitation of the fiber capacity much more, but the NRZ signals offer very high potential especially in terms of simplicity [9].

There are limitations, however, on the performance of optical system due to several effects that are present in optical fibers and amplifiers. Signal propagation through optical fibers can be affected by group velocity dispersion (GVD), polarization mode dispersion (PMD) and the nonlinear effects. The chromatic dispersion that is essentially the GVD when waveguide dispersion is negligible, is a linear effect that introduces pulse broadening generates intersymbol intereference. The PMD arises due the fact that optical fibers for telecommunications have have two polarization modes, in spite of the fact that they are called monomode fibers. These modes have two different group velocities that induce pulse broadening depending on the input signal state of polarization. The transmission impairment due to PMD looks similar to that of the GVD. However, PMD is a random process as compared to the GVD that is a deterministic process. So PMD cannot be controlled at the receiver. Newly installed optical fibers have quite low values of PMD that is about 0.1 $ps/\sqrt{km}$.

The main nonlinear effects that arises in monomode fibers are the Brillouin scattering, Raman scattering and the Kerr effect. Brillouin is a backward scattering that arises from acoustic waves and can generate forward noise at the receiver. Raman scattering is a forward scattering from silica molecules. The Raman gain response is characterized by low gain and wide bandwidth namely about 5 THz. The Raman threshold in conventional fibers is of the order of 500 mW for copolarized pump and Stokes' wave (that is about 1 W for random polarization), thus making Raman effect negligible for a single channel signal. However, it becomes important for multichannel wavelength-division-multiplexed (WDM) signal due to an extremely wide band of wide gain curve.

The Kerr effect of nonlinearity is due to the dependence of the fiber refractive index on the field intensity. This effect mainly manifests as a new frequency when an optical signal propagates through a fiber. In a single channel the Kerr effect induces a spectral broadening and the phase of the signal is modulated according to its power profile. This effect is called self-phase modulation (SPM). The SPM-induced chirp combines with the linear chirp generated by the chromatic dispersion. If the fiber dispersion coefficient is positive namely in the normal dispersion regime, linear and nonlinear chirps have the same sign while in the anomalous dispersion regime they are of opposite signs. In the former case, pulse broadening is enhanced by SPM while in the later case it is reduced. In the anomalous dispersion case the Kerr nonlinearity induces a chirp that can compensate the degradation induced by GVD. Such a compensation is total if soliton signals are used.

If multichannel WDM signals are considered, the Kerr effect can be more degrading since it induces nonlinear cross-talk among the channels that is known as the cross-phase modulation (XPM). In addition WDM generates new frequencies called the Four-Wave mixing (FWM). The other issue in the WDM system is the collision-induced timing jitter that is introduced due to the collision of solitons in different channels. The XPM causes further nonlinear chirp that interacts with the fiber GVD as in the case of SPM. The FWM is a parametric interaction among waves satisfying a particular relationship called phase-matching that lead to power transfer among different channels.

To limit the FWM effect in a WDM it is preferable to operate with a local high GVD that is periodically compensated by devices having an opposite sign of GVD. One such device is a simple optical fiber with opportune GVD and the method is commonly known as the dispersion-management. With this approach the accumulated GVD can be very low and at the same time FWM effect is strongly limited. Through dispersion-management it is possible to acheive highest capacity for both RZ as well as NRZ signals. In that case the overall link dispersion has to be kept very close to zero, while a small amount of chromatic anomalous dispersion is useful for the efficient propagation of a soliton signal. It has been demonstrated that with soliton signals, the dispersion-management is very useful since it reduces collision induced timing jitter [3] and also the pulse interactions. It thus permits the acheivement of higher capacities as compared to the link having constant chromatic dispersion.

2 GOVERNING EQUATIONS

The relevant equation is the Nonlinear Schrödinger's Equation (NLSE) with damping and periodic amplification [1, 7] that is given in the dimensionless form as

$$iu_z + \frac{D(z)}{2}u_{tt} + |u|^2 u = -i\Gamma u + i\left[e^{\Gamma z_a} - 1\right]\sum_{n=1}^{N}\delta(z - nz_a)u \tag{1}$$

Here, Γ is the normalized loss coefficient, z_a is the normalized characteristic amplifier spacing and z and t represent the normalized propagation distance and the normalized time, respectively, expressed in the usual nondimensional units.

Also, $D(z)$ is used to model strong dispersion-management. The fiber dispersion $D(z)$ into two components namely a path-averaged constant value δ_a and a term representing the large rapid variation due to large local values of the dispersion [11, 12]. Thus,

$$D(z) = \delta_a + \frac{1}{z_a}\Delta(\zeta) \tag{2}$$

where $\zeta = z/z_a$. The function $\Delta(\zeta)$ is taken to have average zero (namely $\langle\Delta\rangle = 0$), so that the path-averaged dispersion $\langle D\rangle = \delta_a$. The proportionality factor in front of $\Delta(\zeta)$ is chosen so that both δ_a and $\Delta(\zeta)$ are quantities of order one. In practical situations, dispersion-management is often performed by concatenating together two or more sections of given length of a fiber with different values of fiber dispersion. In the special case of a two-step map it is convenient to write the dispersion map as a periodic extension of [12]

$$\Delta(\zeta) = \begin{cases} \Delta_1 & : \quad 0 \leq |\zeta| < \frac{\theta}{2} \\ \Delta_2 & : \quad \frac{\theta}{2} \leq |\zeta| < \frac{1}{2} \end{cases} \tag{3}$$

where Δ_1 and Δ_2 are given by

$$\Delta_1 = \frac{2s}{\theta} \tag{4}$$

$$\Delta_2 = -\frac{2s}{1-\theta} \tag{5}$$

with the map strength s defined as

$$s = \frac{\theta\Delta_1 - (1-\theta)\Delta_2}{4} \tag{6}$$

Conversely,

$$s = \frac{\Delta_1\Delta_2}{4(\Delta_2 - \Delta_1)} \tag{7}$$

and

$$\theta = \frac{\Delta_2}{\Delta_2 - \Delta_1} \tag{8}$$

Now, taking into account the loss and amplification cycles by looking for a solution of (1) of the form $u(z,t) = A(z)q(z,t)$ for real A and letting A satisfy

$$A_z + \Gamma A - \left[e^{\Gamma z_a} - 1\right] \sum_{n=1}^{N} \delta(z - nz_a)A = 0 \tag{9}$$

it can be shown that (1) transforms to

$$iq_z + \frac{D(z)}{2}q_{tt} + g(z)|q|^2 q = 0 \tag{10}$$

where

$$g(z) = A^2(z) = a_0^2 e^{-2\Gamma(z-nz_a)} \tag{11}$$

for $z \in [nz_a, (n+1)z_a)$ and $n > 0$ and also

$$a_0 = \left[\frac{2\Gamma z_a}{1 - e^{-2\Gamma z_a}}\right]^{\frac{1}{2}} \tag{12}$$

so that $\langle g(z) \rangle = 1$ over each amplification period [12]. Equation (10) governs the propagation of a dispersion-managed soliton through a polarization preserved optical fiber with damping and periodic amplification [14, 18, 19, 20].

3 MULTIPLE CHANNELS

The succesful design of low-loss dispersion-shifted and dispersion-flattened optical fibers with low dispersion over relatively large wavelength range can be used to reduce or completely eliminate the group velocity mismatch for the multi-channel WDM systems resulting in the desirable simultaneous arrival of time aligned bit pulses, thus creating a new class of bit-parallel wavelength links that is used in high speed single fiber computer buses. In spite of the intrinsically small value of the nonlinearity-induced change in the refractive index of fused silica, nonlinear effects in optical fibers cannot be ignored even at relatively low powers. In particular, in WDM systems with simultaneous transmission of pulses of different wavelengths, the cross-phase modulation (XPM) effects needs to be taken into account. Although the XPM will not cause the energy to be exchanged among the different wavelengths, it will lead to the interaction of pulses and thus the pulse positions and shapes gets altered significantly. The multi-channel WDM transmission of co-propagating wave envelopes in a nonlinear optical fiber, including the XPM effect, can be modeled [8] by the following N-coupled NLSE in the dimensionless form

$$iq_z^{(l)} + \frac{D(z)}{2} q_{tt}^{(l)} + g(z)\left\{\left|q^{(l)}\right|^2 + \sum_{m \neq l}^{N} \alpha_{lm}\left|q^{(m)}\right|^2\right\} q^{(l)} = 0 \tag{13}$$

where $1 \leq l \leq N$. Equation (13) is the N-dimensional vector NLSE and is the model for bit-parallel WDM soliton transmission. Here α_{lm} are known as the XPM coefficients. It is well known [7] that the straightforward use of this system for description of WDM transmission could potentially give incorrect results. However, this model can be applied to describe WDM transmission for dispersion flattened fibers, the dispersion of which weakly depends on the operating wavelength.

Another important medium in which the model given by (13) arises is the photorefractive medium [8]. In the case of incoherent beam propagation in a biased photorefractive crystal, which is a noninstantaneous nonlinear media, the diffraction behaviour of that incoherent beam is to be treated somewhat differently. The diffraction behaviour of an incoherent beam can be effectively described by the sum of the intensity contributions from all its coherent components. Then the governing equation of N self-trapped mutually incoherent wave packets in such a media is given by (13).

3.1 INTEGRALS OF MOTION

Equation (13) has at least two integrals of motion and they are the energy (E) and the linear momentum (M) that are respectively given by

$$E = \sum_{l=1}^{N} \int_{-\infty}^{\infty} |q_l|^2 \, dt \tag{14}$$

and

$$M = \frac{i}{2} D(z) \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left(q_l^* \frac{\partial q_l}{\partial t} - q_l \frac{\partial q_l^*}{\partial t} \right) dt \tag{15}$$

The Hamiltonian (H) given by

$$H = \frac{1}{2} \sum_{l=1}^{N} \int_{-\infty}^{\infty} \left\{ D(z) \left| \frac{\partial q_l}{\partial t} \right|^2 - g(z) \sum_{m \neq l}^{N} \alpha_{lm} |q_l|^2 |q_m|^2 \right\} dt \tag{16}$$

is, however, not a conserved quantity unless, in addition to $D(z)$ and $g(z)$ being constants, the matrix of XPM coefficients $\Lambda = (\alpha_{ij})_{N \times N}$ is a symmetric matrix namely $\alpha_{ij} = \alpha_{ji}$ for $1 \leq i, j \leq N$. Thus, for a birefringent fiber the matrix should be of the form

$$\Lambda = \begin{bmatrix} 0 & \alpha_{12} \\ \alpha_{12} & 0 \end{bmatrix} \tag{17}$$

while for a triple channeled fiber

$$\Lambda = \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & 0 & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & 0 \end{bmatrix} \tag{18}$$

and so on. The existence of a Hamiltonian implies that (13) can be written as

$$i\frac{\partial q_l}{\partial z} = \frac{\delta H}{\delta q_l^*} \tag{19}$$

3.2 ASYMPTOTIC ANALYSIS

The fields q_l are expanded in powers of z_a as

$$q_l(\zeta, Z, t) = q_l^{(0)}(\zeta, Z, t) + z_a q_l^{(1)}(\zeta, Z, t) + z_a^2 q_l^{(2)}(\zeta, Z, t) + \cdots \tag{20}$$

Equating coefficients of like powers of z_a gives

$$O\left(\frac{1}{z_a}\right): \quad i\frac{\partial q_l^{(0)}}{\partial \zeta} + \frac{\Delta(\zeta)}{2}\frac{\partial^2 q_l^{(0)}}{\partial t^2} = 0 \tag{21}$$

$$\begin{aligned} O(1): \quad & i\frac{\partial q_l^{(1)}}{\partial \zeta} + \frac{\Delta(\zeta)}{2}\frac{\partial^2 q_l^{(1)}}{\partial t^2} \\ & + \left\{ i\frac{\partial q_l^{(0)}}{\partial Z} + \frac{\delta_a}{2}\frac{\partial^2 q_l^{(0)}}{\partial t^2} + g(z)\left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} \right\} = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} O(z_a): \quad & i\frac{\partial q_l^{(2)}}{\partial \zeta} + \frac{\Delta(\zeta)}{2}\frac{\partial^2 q_l^{(2)}}{\partial t^2} \\ & + \left\{ i\frac{\partial q_l^{(1)}}{\partial Z} + \frac{\delta_a}{2}\frac{\partial^2 q_l^{(1)}}{\partial t^2} + g(z)\left[2\left| q_l^{(1)} \right|^2 q_l^{(1)} + \left(q_l^{(1)} \right)^2 q_l^{(1)*} \right.\right. \\ & \left.\left. + \sum_{m \neq l}^{N} \alpha_{lm} \left\{ 2\left| q_m^{(1)} \right|^2 q_m^{(1)} + \left(q_m^{(1)} \right)^2 q_m^{(1)*} \right\} \right] \right\} = 0 \end{aligned} \tag{23}$$

At $O(1/z_a)$ equation (21), in the Fourier domain, is given by

$$i\frac{\partial \hat{q}_l^{(0)}}{\partial \zeta} - \frac{\omega^2}{2}\Delta(\zeta)\hat{q}_l^{(0)} = 0 \tag{24}$$

whose solution is

$$\hat{q}_l^{(0)}(\zeta, Z, \omega) = \hat{Q}_l^{(0)}(Z, \omega)\, e^{-\frac{i\omega^2}{2}C(\zeta)} \tag{25}$$

where

$$\hat{Q}_l^{(0)}(Z, \omega) = \hat{q}_l^{(0)}(0, Z, \omega) \tag{26}$$

At $O(1)$, equation (22) is solved in the Fourier domain by substituting the solution given by (25) into (22). This gives

$$\begin{aligned} & i\frac{\partial \hat{q}_l^{(1)}}{\partial \zeta} - \frac{\omega^2}{2}\Delta(\zeta)\hat{q}_l^{(1)} \\ & \quad = -e^{-\frac{i\omega^2}{2}C(\zeta)}\left(\frac{\partial \hat{Q}_l^{(0)}}{\partial Z} - \frac{\omega^2}{2}\delta_a \hat{Q}_l^{(0)}\right) - g(\zeta)\int_{-\infty}^{\infty}\left(\left|q_l^{(0)}\right|^2 + \sum_{m\neq l}^{N}\alpha_{lm}\left|q_m^{(0)}\right|^2\right) q_l^{(0)} e^{i\omega t} dt \end{aligned} \tag{27}$$

Equation (27) is an inhomogenous equations for $\hat{q}_l^{(1)}$ and with the homogenous parts having the same structure as in (21). For the non-secularity condition of $\hat{q}_l^{(1)}$, FA gives the condition on $\hat{Q}_l^{(0)}(Z,\omega)$ as

$$\begin{aligned} & \frac{\partial \hat{Q}_l^{(0)}}{\partial Z} - \frac{\omega^2}{2}\delta_a \hat{Q}_l^{(0)} \\ & \quad + \int_0^1\int_{-\infty}^{\infty} e^{\frac{i\omega^2}{2}C(\zeta)} g(\zeta)\left(\left|q_l^{(0)}\right|^2 + \sum_{m\neq l}^{N}\alpha_{lm}\left|q_m^{(0)}\right|^2\right) q_m^{(0)} e^{i\omega t} dt d\zeta = 0 \end{aligned} \tag{28}$$

Equation (28) can be simplified to

$$\begin{aligned} & \frac{\partial \hat{Q}_l^{(0)}}{\partial Z} - \frac{\omega^2}{2}\delta_a \hat{Q}_l^{(0)} \\ & \quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_0\left(\omega_1\omega_2\right)\hat{Q}_l^{(0)}\left(Z,\omega_1+\omega_2\right)\Big[\hat{Q}_l^{(0)}\left(Z,\omega+\omega_1\right)\hat{Q}_l^{(0)}\left(Z,\omega+\omega_1+\omega_2\right) \\ & \quad + \sum_{m\neq l}^{N}\alpha_{lm}\hat{Q}_m^{(0)}\left(Z,\omega+\omega_1\right)\hat{Q}_m^{(0)}\left(Z,\omega+\omega_1+\omega_2\right)\Big] d\omega_1 d\omega_2 = 0 \end{aligned} \tag{29}$$

where the kernel $r_0(x)$ is given by

$$r_0(x) = \frac{1}{(2\pi)^2}\int_0^1 g(\zeta)e^{iC(\zeta)x}\, dx \tag{30}$$

Equation (29) is commonly known as GTE for the propagation of solitons through multiple channels [7]. Equation (22) will now be solved to obtain $q_l^{(1)}(\zeta,Z,t)$. Substituting $\hat{Q}_l^{(0)}$ into the right side of equation (27) and using (24) and (28) gives

$$\frac{\partial}{\partial \zeta}\left[i\hat{q}_l^{(1)}e^{\frac{i\omega^2}{2}C(\zeta)}\right]$$

$$
\begin{aligned}
&= \int_0^1 \int_{-\infty}^{\infty} g(\zeta) e^{\frac{i\omega^2}{2}C(\zeta)} \left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt d\zeta \\
&- g(\zeta) e^{\frac{i\omega^2}{2}C(\zeta)} \int_{-\infty}^{\infty} \left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt
\end{aligned} \tag{31}
$$

which integtrates to

$$
\begin{aligned}
& i\hat{q}_l^{(1)} e^{\frac{i\omega^2}{2}C(\zeta)} = \hat{Q}_l^{(1)}(Z, \omega) \\
& \quad + \zeta \int_0^1 \int_{-\infty}^{\infty} g(\zeta) e^{\frac{i\omega^2}{2}C(\zeta)} \left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt d\zeta \\
& \quad - \int_0^{\zeta} \int_{-\infty}^{\infty} g(\zeta') e^{\frac{i\omega^2}{2}C(\zeta')} \left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt d\zeta'
\end{aligned} \tag{32}
$$

where

$$
\hat{Q}_l^{(1)}(Z, \omega) = i\hat{q}_l^{(1)}(0, Z, \omega) e^{\frac{i\omega^2}{2}C(0)} \tag{33}
$$

Also, $\hat{Q}_l^{(1)}(Z, \omega)$ is so chosen that

$$
\int_0^1 i\hat{q}_l^{(1)} e^{\frac{i\omega^2}{2}C(\zeta)} d\zeta = 0 \tag{34}
$$

which is going to be an useful relation for subsequent orders. Applying (34) to (32) gives

$$
\begin{aligned}
& \hat{Q}_l^{(1)}(Z, \omega) = \\
& \int_0^1 \int_0^{\zeta} \int_{-\infty}^{\infty} g(\zeta') e^{\frac{i\omega^2}{2}C(\zeta')} \left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt d\zeta' d\zeta \\
& - \frac{1}{2} \int_0^1 \int_{-\infty}^{\infty} g(\zeta) e^{\frac{i\omega^2}{2}C(\zeta)} \left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt d\zeta
\end{aligned} \tag{35}
$$

Now, (32), by virtue of (35), can be written as

$$
\begin{aligned}
& \hat{q}_l^{(1)}(\zeta, Z, \omega) = \\
& ie^{\frac{i\omega^2}{2}C(\zeta)} \left[\int_0^{\zeta} \int_{-\infty}^{\infty} g(\zeta') e^{\frac{i\omega^2}{2}C(\zeta')} \left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt d\zeta' \right. \\
& - \int_0^1 \int_0^{\zeta} \int_{-\infty}^{\infty} g(\zeta') e^{\frac{i\omega^2}{2}C(\zeta')} \left(\left| q_m^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt d\zeta' d\zeta \\
& \left. - \left(\zeta - \frac{1}{2} \right) \int_0^1 \int_{-\infty}^{\infty} g(\zeta) e^{\frac{i\omega^2}{2}C(\zeta)} \left(\left| q_l^{(0)} \right|^2 + \sum_{m \neq l}^{N} \alpha_{lm} \left| q_m^{(0)} \right|^2 \right) q_l^{(0)} e^{i\omega t} dt d\zeta \right]
\end{aligned} \tag{36}
$$

which can be further rewritten as

$$\begin{aligned}
&\hat{q}_l^{(1)}(\zeta, Z, \omega) = \\
&\quad ie^{-\frac{i\omega^2}{2}C(\zeta)}\Bigg[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{Q}_l^{(0)*}\left(\omega+\Omega_1+\Omega_2\right)\Big\{\hat{Q}_l^{(0)}\left(\omega+\Omega_1\right)\hat{Q}_l^{(0)}\left(\omega+\Omega_2\right) \\
&\quad +\sum_{m\neq l}^{N}\alpha_{lm}\hat{Q}_l^{0)}\left(\omega+\Omega_1\right)\hat{Q}_l^{(0)}\left(\omega+\Omega_2\right)\Bigg\} \\
&\quad \Bigg\{\int_0^{\zeta} g(\zeta')e^{i\Omega_1\Omega_2 C(\zeta')}d\zeta' - \int_0^1\int_0^{\zeta} g(\zeta')e^{i\Omega_1\Omega_2 C(\zeta')}d\zeta' d\zeta \\
&\quad -\left(\zeta-\frac{1}{2}\right)\int_0^1 g(\zeta)e^{i\Omega_1\Omega_2 C(\zeta)}d\zeta\Bigg\}\,d\Omega_1\Omega_2\Bigg]
\end{aligned} \tag{37}$$

Thus, at $O(z_a)$,

$$\hat{q}_l(\zeta, Z, \omega) = \hat{q}_l^{(0)}(\zeta, Z, \omega) + z_a\hat{q}_l^{(1)}(\zeta, Z, \omega) \tag{38}$$

Moving on to the next order at $O(z_a^2)$, one can note that the GTE given by (29) is allowed to have an additional term of $O(z_a)$ as

$$\begin{aligned}
&\frac{\partial\hat{Q}_l^{(0)}}{\partial Z} - \frac{\omega^2}{2}\delta_a\hat{Q}_l^{(0)} \\
&\quad +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_0\left(\omega_1\omega_2\right)\hat{Q}_l^{(0)}\left(Z,\omega_1+\omega_2\right)\Big[\hat{Q}_l^{(0)}\left(Z,\omega+\omega_1\right)\hat{Q}_l^{(0)}\left(Z,\omega+\omega_1+\omega_2\right) \\
&\quad +\sum_{m\neq l}^{N}\alpha_{lm}\hat{Q}_m^{(0)}\left(Z,\omega+\omega_1\right)\hat{Q}_m^{(0)}\left(Z,\omega+\omega_1+\omega_2\right)\Bigg]d\omega_1 d\omega_2 = z_a\hat{n}_l(Z,\omega) + O(z_a^2)
\end{aligned} \tag{39}$$

The higher order correction $\hat{n}_l$ can be obtained from suitable a non-secular conditions at $O(z_a^2)$ in (23). Now, equation (23), in the Fourier domain, is

$$\begin{aligned}
&\frac{\partial}{\partial\zeta}\left[i\hat{q}_l^{(2)}e^{\frac{i\omega^2}{2}C(\zeta)}\right] + \hat{n}_l + e^{\frac{i\omega^2}{2}C(\zeta)}\left(i\frac{\partial\hat{q}_l^{(1)}}{\partial Z} - \frac{\omega^2}{2}\delta_a\hat{q}_l^{(1)}\right) \\
&\quad +e^{\frac{i\omega^2}{2}C(\zeta)}g(\zeta)\int_{-\infty}^{\infty}\Bigg[2\left|q_l^{(0)}\right|^2 q_l^{(1)} + \left(q_l^{(0)}\right)^2 q_l^{(1)*} \\
&\quad +\sum_{m\neq l}^{N}\alpha_{lm}\left\{2\left|q_m^{(0)}\right|^2 q_m^{(1)} + \left(q_m^{(0)}\right)^2 q_m^{(1)*}\right\}\Bigg]e^{i\omega t}dt = 0
\end{aligned} \tag{40}$$

But, again (33) gives

$$\int_0^1 \hat{q}_l^{(1)}e^{\frac{i\omega^2}{2}C(\zeta)}d\zeta = 0 \tag{41}$$

Applying the non-secularity condition (40) to (39) gives

$$\hat{n}_l = -\int_0^1 \int_{-\infty}^{\infty} e^{\frac{i\omega^2}{2}C(\zeta)} g(\zeta)$$
$$\left[2\left|q_l^{(0)}\right|^2 q_l^{(1)} + \left(q_l^{(0)}\right)^2 q_l^{(1)*} + \sum_{m\neq l}^{N} \alpha_{lm}\left\{2\left|q_m^{(0)}\right|^2 q_m^{(1)} + \left(q_m^{(0)}\right)^2 q_m^{(1)*}\right\}\right] e^{i\omega t} dt d\zeta \tag{42}$$

Using (25) and (35), equation (41) can be written as

$$\hat{n}_l = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_1\left(\omega_1\omega_2, \Omega_1\Omega_2\right)$$
$$\left[\left\{2\hat{Q}_l^{(0)}\left(\omega+\omega_1\right)\hat{Q}_l^{(0)*}\left(\omega+\omega_1+\omega_2\right)\hat{Q}_l^{(0)}\left(\omega+\omega_2+\Omega_1\right)\right.\right.$$
$$\hat{Q}_l^{0)}\left(\omega+\omega_2+\Omega_2\right)\hat{Q}_l^{(0)*}\left(\omega+\omega_2+\Omega_1+\Omega_2\right)$$
$$-\hat{Q}_l^{(0)}\left(\omega+\omega_1\right)\hat{Q}_l^{(0)}\left(\omega+\omega_2\right)\hat{Q}_l^{(0)*}\left(\omega+\omega_1+\omega_2+\Omega_1\right)$$
$$\left.\hat{Q}_l^{(0)*}\left(\omega+\omega_1+\omega_2-\Omega_2\right)\hat{Q}_l^{(0)*}\left(\omega+\omega_1+\omega_2+\Omega_1-\Omega_2\right)\right\}$$
$$+\sum_{m\neq l}^{N}\alpha_{lm}\left\{2\hat{Q}_m^{(0)}\left(\omega+\omega_1\right)\hat{Q}_m^{(0)*}\left(\omega+\omega_1+\omega_2\right)\hat{Q}_m^{(0)}\left(\omega+\omega_2+\Omega_1\right)\right.$$
$$\hat{Q}_m^{(0)}\left(\omega+\omega_2+\Omega_2\right)\hat{Q}_m^{(0)*}\left(\omega+\omega_2+\Omega_1+\Omega_2\right)$$
$$-\hat{Q}_m^{(0)}\left(\omega+\omega_1\right)\hat{Q}_m^{(0)}\left(\omega+\omega_2\right)\hat{Q}_m^{(0)*}\left(\omega+\omega_1+\omega_2+\Omega_1\right)$$
$$\left.\left.\hat{Q}_m^{(0)*}\left(\omega+\omega_1+\omega_2-\Omega_2\right)\hat{Q}_m^{(0)*}\left(\omega+\omega_1+\omega_2+\Omega_1-\Omega_2\right)\right\}\right] d\omega_1 d\omega_2 d\Omega_1 d\Omega_2 \tag{43}$$

where, the kernel $r_1(x, y)$ is given by

$$r_1(x,y) = \frac{1}{(2\pi)^4}\int_0^1\int_0^{\zeta} g(\zeta)g(\zeta')e^{i(xC(\zeta)+yC(\zeta))}d\zeta d\zeta'$$
$$-\left[\int_0^1 g(\zeta)e^{ixC(\zeta)}d\zeta\right]\left[\int_0^{\zeta} g(\zeta')e^{ixC(\zeta')}d\zeta'\right]$$
$$-\left[\int_0^1\left(\zeta-\frac{1}{2}\right)g(\zeta)e^{ixC(\zeta)}d\zeta\right]\left[\int_0^1 g(\zeta)e^{iyC(\zeta)}d\zeta\right] \tag{44}$$

Equation (42) represents the HO-GTE for the propagation of solitons through multiple channels.

4 PROPERTIES OF THE KERNEL

The HO-GTE, for different types of fibers, are the fundamental equations that govern the evolution of optical pulses for a strong dispersion-managed soliton systems corresponding to the frequency and time domain respectively. In these GT equations, all the fast variations and large quantities are removed and so they contain only slowly varying quantities of order one. These equations are not limited to the case $\delta_a > 0$, however, they are also applicable to the case of pulse dynamics with zero or normal values of average dispersion. If the fiber dispersion is constant namely if $\Delta(\zeta) = 0$, then $C(\zeta) = 0$ and so $r_0(x) = 1/\left(2\pi\right)^2$ The kernels $r_0(x)$ and $r_1(x,y)$ are now going to be studied in the following two cases

4.1 Lossless Case

For the lossless case, namely when $g(\zeta) = 1$, the kernels $r_0(x)$ and $r_1(x,y)$ for a two-step map defined in (3) take very simple forms namely

$$r_0(x) = \frac{1}{\left(2\pi\right)^2}\frac{\sin(sx)}{sx} \tag{45}$$

$$\begin{aligned} r_1(x,y) = {} & \frac{i(2\theta-1)}{2s^3x^2y^2(x+y)} \\ & \Big[sxy\left\{y\cos(sx)\sin(sy) - x\cos(sy)\sin(sx)\right\} + \left(x^2-y^2\right)\sin(sx)\sin(sy)\Big] \end{aligned} \tag{46}$$

It can be seen that θ appears in r_1 but not in r_0. This means that the leading order of HO-GTE is independent of θ. Equation (44) also shows that $r_1(x,y)$ vanishes at $\theta = 1/2$ so that the leading order GTE is valid for long distances $O(1/z_a)$ if the positive and negative dispersion lengths of the fiber are the same. It can also be observed that

$$\lim_{s\to 0} r_0(x) = \frac{1}{\left(2\pi\right)^2} \tag{47}$$

$$\lim_{s\to 0} r_1(x,y) = 0 \tag{48}$$

This shows that the higher-orde GTE reduces to the ideal NLSE as the map strength approaches zero.

4.2 Lossy Case

For the lossy case, namely when $g(\zeta) \neq 1$, the kernel $r_0(x)$ depends on the relative position of the amplifier with respect to the dispersion map. The two step map given by $\Delta(\zeta)$ in (3) is considered and ζ_a is defined to represent the position of the amplifier within the dispersion map. So $|\zeta_a| < 1/2$ and $\zeta_a = 0$ means that the amplifier is placed at the mid point of the anomalous fiber segment. The function $g(\zeta)$ given by (11) can then be written as

$$g(\zeta) = \frac{2Ge^{2G}}{\sinh(2G)} e^{-4G(\zeta - n\zeta_a)} \tag{49}$$

for $\zeta_a + n \leq \zeta < \zeta_a + n + 1$ where $G = \Gamma z_a/2$. The kernel $r_0(x)$ in the lossy case is computed in a similar method as in the lossless case. If $|\zeta_a| < \theta/2$, namely the amplifier is located in the anomalous fiber segment, the resulting expression for kernel is

$$\begin{aligned} r_0(x) = {} & \frac{1}{(2\pi)^2} \frac{Ge^{iC_0 x}}{(sx + 2iG\theta)(sx - 2iG(1-\theta))} \\ & \left[e^{G(4\zeta_a - 2\theta + 1)} sx \frac{\sin(sx - 2iG(1-\theta))}{\sinh(2G)} + i\theta e^{i(4\zeta_a - 2\theta + 1)\frac{sx}{2\theta}} (sx - 2iG(1-\theta)) \right] \end{aligned} \tag{50}$$

In (48), unlike the lossless case, the kernel $r_0(x)$ is complex and is explicitly dependent on the parameters θ, Γ, z_a and ζ_a in a nontrivial way. However, one still gets

$$\lim_{s \to 0} r_0(x) = \frac{1}{(2\pi)^2} \tag{51}$$

and moreover

$$\lim_{G \to 0} r_0(x) = \frac{1}{(2\pi)^2} \frac{\sin(sx)}{sx} \tag{52}$$

which means that as $z_a \to 0$, (45) is recovered. For the particular case $\theta = 1/2$, $\zeta_a = 0$ which corresponds to fiber segments of equal length with amplifiers placed at the middle of the anomalous fiber segment, the kernel modifies to

$$r_0(x) = \frac{1}{(2\pi)^2} \frac{G}{x^2 s^2 + G^2} \left[sx \frac{\sin(sx)}{\sinh(G)} + isx \left\{ 1 - \frac{\cos(sx)}{\cosh(G)} \right\} + G \right] \tag{53}$$

Also for $g(\zeta) \neq 1$, (44) gives

$$\lim_{s \to 0} r_1(x, y) = 0 \tag{54}$$

Thus, even in the lossy case, HO-GTE reduces to the case of ideal NLSE.

5 CONCLUSIONS

In this paper, the dynamics of optical solitons, propagating through DWDM systems, with strong dispersion-management, was studied. The technique that was used is the multiple-scale perturbation expansion. By using this technique the pulses in the Fourier domain was decomposed into a slowly evolving amplitude and a rapid phase that descrides the chirp of the pulse. The fast phase is calculated explicitly that is driven by the large variations of the dispersion about the average. The amplitude evolution is described by the nonlocal evolution equations that is the HO-GTE.

These equations can be used to study the propagation of solitons with higher order accuracy, namely accuracy to $O(z_a^2)$. Also, the dynamics of quasilinear pulses [2, 3, 8], in optical fibers, can also be studied with greater accuracy. HO-GTE can also be used to study the four-wave mixing, timing and amplitude jitter and ghost pulses [5] for optical fibers, with better estimates and further accuracy than that was already obtained before. Better yet, HO-GTE can be used to study the detailed asymptotic properties governing the long-scale dynamics of optical pulses. It needs to be noted here that the derivation of the HO-GTE is valid for any arbitrary dispersion map $D(z)$ and with the general effects of damping and periodic amplification $g(z)$.

Although the HO-GTE is useful for studying the structure and properties, it is inconvenient for numerical computations because of the presence of the four-fold integrals that are given in the $O(z_a)$ terms of the HO-GTE. In the case of polarization preserving fibers, there was some numerical analysis done with special solutions of the HO-GTE and bi-solitons, tri-solitons and quartic-solitons was observed [4, 16, 17].

In future, one can extend this study to include the perturbation terms, for example, filters, higher order dispersion, Raman scattering, self-steepening just to name a few. Also, it is possible to take a look at the GTE and HO-GTE in the context of other laws of nonlinearity like parabolic law, saturable law and others.

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