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Strong Convergence of CQ Iteration for Asymptotically Nonexpansive Mappings

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Abstract. Tae-Hwa Kim and Hong-Kun Xu [Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear Analysis 64(2006)1140-1152] proved the strong convergence theorems of modified Mann iterations for asymptotically nonexpansive mappings and semigroups on bounded subset C of a Hilbert space by the CQ iteration method. The purpose of this paper is to modify the CQ iteration scheme of Tae-Hwa Kim and Hong-Kun Xu, and to prove strong convergence theorems for asymptotically nonexpansive mappings on subset C of a Hilbert space without the condition of boundedness of subset C . The convergence rate of iteration process presented in this paper is faster than the iteration process of Tae-Hwa Kim and Hong-Kun Xu. In this article, the interesting method of proof is used which without use the demi-closedness principle.

Keywords: Strong convergence; CQ method; Asymptotically nonexpansive mapping.

1. INTRODUCTION

Let X be a real Banach space, C a nonempty closed convex subset of X, and T: $C \to C$ a mapping. Recall that T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [4] if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all integers $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$.

Construction of fixed points of nonexpansive mappings (and asymptotically nonexpansive mappings) is an important subject in the theory of nonexpansive mappings and finds application in a number of applied areas, in particular, in image recovery and signal processing (see,e.g.,[2,8,11,16,17]). However, the sequence $\{T^n x\}_{n=0}^{\infty}$ of iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus averaged iterations prevail. Indeed, Mann's iterations do have weak convergence. More precisely, a Mann's iteration procedure is a sequence $\{x_n\}$ which is generated in the following recursive way:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily. For example, Reich [9] proved that if X is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However we note that Mann's iterations have only weak convergence even in a Hilbert space [3].

Attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi[7] proposed the following modification of Mann iteration method (1.1) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H.

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$.

The adaptation of Mann's iteration (1.1) to asymptotically nonexpansive mappings T is given below

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \quad n \geq 0. \tag{1.3}$$

Weak convergence of the sequence $\{x_n\}$ generated by (1.3) is proved by Schu [10] (see also Tan and Xu [14]).

Attempts to modify the Mann iteration method (1.3) so that strong convergence is guaranteed have recently been made In 2006,T.H.Kim and H.K.Xu [18] proposed the following modification of the Mann iteration method (1.3) for asymptotically non-expansive mapping on a bounded closed convex subset C

in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.4}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0 \ \ as\ n \to \infty$.

T.H.Kim and H.K.Xu proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}(x_0)$.

It is purpose of this paper to modify iteration process (1.4) for asymptotically nonexpansive mappings on a closed convex subset C which without boundedness in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ chosen\ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, n \ge 1 \\ C_0 = \{z \in C : \|y_0 - z\|^2 \le \|x_0 - z\|^2 + \theta_0\}, \\ Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, n \ge 1 \\ Q_0 = C \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.5}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\sup_{x \in A} \|x_n - x\|)^2,$$
$$A = \{\ y \in F(T) : \|y - p_0\| \le 1\},\ p_0 = P_{F(T)}(x_0).$$

We shall prove that iteration sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}(x_0)$ provided the sequence $\{\alpha_n\}$ is bounded from above one.

2. MAIN RESULTS

In 2006, Tae-Hwa Kim and Hong-Kun Xu[18] proved the following theorem.

Theorem 2.1[18] *Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequences in (0,1) such that $\alpha_n \le a$, for all n and for some $0 < a < 1$. Define a sequence $\{x_n\}$ in C by the algorithm (1.4). Then $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$.*

The result of this paper is following.

Theorem 2.2 *Let T be an asymptotically nonexpansive mapping with nonempty fixed points set $F(T)$ defined on a closed convex subset C of a Hilbert space H.*

Assume that $\{\alpha_n\}$ *is a sequences in [0,1] such that* $\alpha_n \leq a$, *for all* n *and for some* $a \in (0,1)$. *Define a sequence* $\{x_n\}$ *in* C *by the algorithm (1.5). Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}(x_0)$.

Proof We observe that C_n is convex. Indeed, the defining inequality in C_n is equivalent to the following inequality

$$\langle 2(x_n - y_n), z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \theta_n$$

which is affine(and hence convex) in z.

It is obvious that

$$A = \{\ y \in F(T) : \|y - p_0\| \leq 1\},$$

is bounded closed convex subset of H, where $p_0 = P_{F(T)}(x_0)$.

Now, we show that $A \subset C_n$ for all n. Indeed, for any $p \in A$ we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T^n x_n - p)\| \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2\|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 + (1-\alpha_n)(k_n^2 - 1)\|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 + (1-\alpha_n)(k_n^2 - 1)(\sup_{x\in A}\|x_n - x\|)^2 \\ &\leq \|x_n - p\|^2 + \theta_n. \end{aligned} \tag{2.1}$$

So $p \in C_n$ for all n, that is $A \subset C_n$ for all n.

Next we show that $A \subset C_n \bigcap Q_n$, for all $n \geq 0$. It suffices to show that $A \subset Q_n$, for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $A \subset F(T) \subset Q_0$. Assume that $A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0, \quad \forall z \in Q_n \cap C_n,$$

as $A \subset C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in A$. This together with the definition of Q_{n+1} implies that $A \subset Q_{n+1}$. Hence the $A \subset C_n \bigcap Q_n$ holds for all n.

By the definition of Q_n, we known that Q_n is convex and $x_n \in Q_n$ so that $x_n = P_{Q_n}(x_0)$ which together with the fact that $x_{n+1} \in C_n \bigcap Q_n \subset Q_n$ implies that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|. \tag{2.2}$$

This shows that the sequence $\{\|x_n - x_0\|\}$ is increasing.

Now, we show that $\{x_n\}$ is bounded sequence. Indeed, since $A \subset Q_n, n \geq 0$, then for any $z \in A$, we have $\langle x_n - z, x_0 - x_n\rangle \geq 0$ which implies that

$$\|x_0 - z\|^2 = \|x_n - z + x_0 - x_n\|^2$$

$$= \|x_n - z\|^2 + \|x_0 - x_n\|^2 + 2\langle x_n - z, x_0 - x_n\rangle$$

$$\geq \|x_n - z\|^2 + \|x_0 - x_n\|^2. \tag{2.3}$$

It follows from above inequality (2.3) that $\{x_n\}$ is bounded which together with the inequality (2.2) implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists .

Noticing again that $x_n = P_{Q_n}(x_0)$ and for any positive integer m we have $x_{n+m} \in Q_{n+m-1} \subset Q_n$ which implies that $\langle x_{n+m} - x_n, x_n - x_0\rangle \geq 0$, and noticing the identity

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle, \quad \forall u, v \in H.$$

we have

$$\|x_{n+m} - x_n\|^2 = \|(x_{n+m} - x_0) - (x_n - x_0)\|^2$$

$$\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle$$

$$\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0,\ n \to \infty. \tag{2.4}$$

From result (2.4) we known that $\{x_n\}$ is a Cauchy sequence in C, so that there exists a point $p \in C$ such that $\lim_{n\to\infty} x_n = p$

We now prove that $\|Tx_n - x_n\| \to 0$, indeed,

$$\|T^n x_n - x_n\| = \frac{1}{1 - \alpha_n}\|y_n - x_n\|$$

$$\leq \frac{1}{1 - \alpha_n}(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|).$$

Since $x_{n+1} \in C_n$, then

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n.$$

Because $\theta_n \to 0$, and we have proved that $\|x_{n+1} - x_n\| \to 0$, so that $\|y_n - x_{n+1}\| \to 0$, therefore, which leads to

$$\|T^n x_n - x_n\| \to 0. \tag{2.5}$$

Putting $k = \sup\{k_n : n \geq 1\} < \infty$, we deduce that

$$\|Tx_n - x_n\| \leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\|$$

$$+\|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|$$

$$\leq k\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1 + k)\|x_{n+1} - x_n\| \to 0. \tag{2.6}$$

We have proved that $\{x_n\}$ converges in norm to a point $p \in C$ which together with the (2.6) implies that p is a fixed point of T.

We claim that $p = z_0 = P_{F(T)}(x_0)$*, if not, we have that* $\|x_0 - p\| > \|x_0 - z_0\|$*. There must exists a positive integer* N*, if* $n > N$ *then* $\|x_0 - x_n\| > \|x_0 - z_0\|$ *which leads to*

$$\|z_0 - x_0\|^2 = \|z_0 - x_n + x_n - x_0\|^2$$

$$= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ *which implies that* $z_0 \overline{\in} Q_n$*, so that* $z_0 \overline{\in} F(T)$*, this is a contradiction. This completes the proof.*

Remark In this article, the interesting method of proof is used which without use the demi-closedness principle. On the other hand, from the definitions of Q_n and C_n, it is easy to see that, Q_n and C_n are monotone decreasing sets, so that the convergence rate of iteration process (1.5) presented in this paper is faster than the iteration process of Tae-Hwa Kim and Hong-Kun Xu[18]. The result of this paper is proved without the condition of boundedness of subset C.

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