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Identification of Some Real Interpolation of Quasi-Banach Spaces

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Abstract

We inter relate the real interpolation space with the quasi-Banach couple (A_0, A_1), $(A_0 + A_1, A_1)$, $(A_0, A_0 \cap A_1)$, and $(A_0 + A_1, A_0 \cap A_1)$ that A_j is c_j normed. Proving among others the identities

$$(A_0 + A_1, A_0)_{\theta,q} \cap (A_0 + A_1, A_1)_{\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\theta,q}$$
$$(A_0 + A_1, A_0)_{\theta,q} \cap (A_1, A_0 + A_1)_{\theta,q} = (A_0, A_1)_{\theta,q}$$
$$(A_0, A_0 \cap A_1)_{\theta,q} + (A_1, A_0 \cap A_1)_{\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\theta,q}$$
$$(A_0, A_0 \cap A_1)_{\theta,q} + (A_0 \cap A_1, A_1)_{\theta,q} = (A_0, A_1)_{\theta,q}$$

for all $0 < q \leq \infty$, $0 < \theta < 1$, and $c_1/c_0 \leq 1$.

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1 Introduction

Our main reference to the theory of interpolation space is [2]. Let $\bar{A} = (A_0, A_1)$ be a quasi-Banach couple, let $0 < \theta < 1$ and $0 < q \leq \infty$. The real interpolation space $(A_0, A_1)_{\theta,q}$ consist of all elements $a \in A_0 + A_1$ having a finite quasi-norm

$$\|a\|_{\theta,q,} = \begin{cases} (\sum_{\nu\in Z}(2^{-\nu\theta}K(2^\nu, a))^q)^{1/q} & if\ 0 < q < \infty \\ sup_{\nu\in Z}\{2^{-\nu\theta}K(2^\nu, a)\} & if\ q = \infty \end{cases} .$$

Here, for $0 < t < \infty$, we put

$$K(t, a) = K(t, a; A_0, A_1) = inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \ : \ a = a_0 + a_1, a_j \in A_j\}.$$

For $0 < \theta < 1$ we abbreviate $\bar{\theta} = max(\theta, 1 - \theta)$ and $\underline{\theta} = min(\theta, 1 - \theta)$.

2 Main result

In the following (A_0, A_1) will always denote a quasi-Banach couple that A_j is c_j normed with $c_1/c_0 \leq 1$.

Theorem 2.1 *Let (A_0, A_1) be a quasi-Banach couple and $a \in A_0 + A_1$. Then*

$$K(t, a; A_0, A_1) = K(t, a; A_0, A_0 + A_1) \quad (t \geq 1).$$

Proof.
Let $a = a_0 + a_1$ with $a_0 \in A_0, a_1 \in A_0 + A_1$ and let $a_1 = b_0 + b_1$ with $b_0 \in A_0, b_1 \in A_1$. Then $K(t, a; A_0, A_1) \leq \|a_0 + b_0\|_{A_0} + t\|b_1\|_{A_1} \leq c_0(\|a_0\|_{A_0} + \|b_0\|_{A_0}) + t\|b_1\|_{A_1} \leq C_0(\|a_0\|_{A_0} + t[\|b_0\|_{A_0} + \|b_1\|_{A_1}])$ for $t \geq 1$ and $c_0 \geq 1$. Taking the infimum with respect to the decomposition $a_1 = b_0 + b_1$ we obtain $K(t, a; A_0, A_1) \leq C_0(\|a_0\|_{A_0} + t\|a_1\|_{A_0+A_1})$, and going over to the infimum again yields $K(t, a; A_0, A_1) \leq C_0K(t, a; A_0, A_0 + A_1)$. The reverse inequality $K(t, a; A_0, A_0 + A_1) \leq C_0K(t, a; A_0, A_1)$ holds trivially for all $t > 0$.

Recall that (X,Y) is a quasi-Banach couple and $X \subset Y$, then for having $a \in (X, Y)_{\theta,q}$ only the behaviour of $K(t, a; X, Y)$ on $(1, \infty)$ is relevant and $(\sum_{\nu\in Z}(2^{-\nu\theta}K(2^\nu, a))^q)^{1/q}$, $\nu \geq 0$ is an equivalent quasi-norm on $(X, Y)_{\theta,q}$. An analogous remark applies in the case $Y \subset X$.

Theorem 2.2 *Let (A_0, A_1) be a quasi-Banach couple. Then*

$$(A_0, A_0 + A_1)_{\theta,q} = \{a \in A_0 + A_1|(\sum_{\nu\in Z}(2^{-\nu\theta}K(2^\nu, a))^q)^{1/q}, \ \nu \geq 0\}$$

$$(A_0 + A_1, A_1)_{\theta,q} = \{a \in A_0 + A_1|(\sum_{\nu\in Z}(2^{-\nu\theta}K(2^\nu, a))^q)^{1/q}, \ \nu \leq 0\}.$$

Proof.
This is a straightforward application of theorem 2.1 and the remarks preceding this theorem.

Theorem 2.3 *Let (A_0, A_1) be a quasi-Banach couple and $a_0 \in A_0$. Then*

$$K(t, a_0; A_0, A_1) \le K(t, a_0; A_0, A_0 \cap A_1) \qquad (t > 0).$$

$$K(t, a_0; A_0, A_0 \cap A_1) \le (c_0 + 1)K(t, a_0; A_0, A_1) + c_0 t\|a_0\|_{A_0} \qquad (t \le 1).$$

Proof.
The first assertion holds trivially. To prove the second, take $a_0 = b_0 + b_1$, where $b_0 \in A_0, b_1 \in A_1$. Then obviously $b_1 \in A_0 \cap A_1$ and

$$\begin{aligned} K(t, a_0; A_0, A_0 \cap A_1) &\le \|b_0\|_{A_0} + t\|b_1\|_{A_0 \cap A_1} \\ &\le \|b_0\|_{A_0} + t\|b_1\|_{A_0} + t\|b_1\|_{A_1} \\ &\le \|b_0\|_{A_0} + t\|b_1\|_{A_1} + c_0 t\|a_0\|_{A_0} + c_0 t\|b_0\|_{A_0} \\ &\le (c_0 + 1)\|b_0\|_{A_0} + t\|b_1\|_{A_1} + c_0 t\|a_0\|_{A_0} \end{aligned}$$

Taking the infimum yields $K(t, a_0; A_0, A_0 \cap A_1) \le (c_0 + 1)K(t, a_0; A_0, A_1) + c_0 t\|a_0\|_{A_0}$.

Lemma 2.4 *Let (A_0, A_1) be a quasi-Banach couple. Then the following identities hold.*

$$(A_0 + A_1, A_1)_{\theta,q} \cap A_0 = (A_0, A_1)_{\theta,q} \cap A_0 = (A_0, A_0 \cap A_1)_{\theta,q}. \tag{1}$$

$$(A_0 \cap A_1, A_1)_{\theta,q} + A_0 = (A_0, A_1)_{\theta,q} + A_0 = (A_0, A_0 + A_1)_{\theta,q}. \tag{2}$$

Proof.
Let us prove the identity (1). The chain of inclusions " $\supset$ " is clear, whence we have to show $(A_0 + A_1, A_1)_{\theta,q} \cap A_0 \subset (A_0, A_0 \cap A_1)_{\theta,q}$. Take $a_0 \in (A_0 + A_1, A_1)_{\theta,q} \cap A_0$. Since $a_0 \in A_0$, only the behaviour of $K(t, a_0; A_0, A_0 \cap A_1)$ on (0,1) matters. According to theorem 2.3 and theorem 2.1

$$\begin{aligned} K(t, a_0; A_0, A_0 \cap A_1) &\le (c_0 + 1)K(t, a_0; A_0, A_1) + c_0 t\|a_0\|_{A_0} \\ &= (c_0 + 1)tK(t^{-1}, a_0; A_1, A_0) + c_0 t\|a_0\|_{A_0} \\ &= (c_0 + 1)tK(t^{-1}, a_0; A_1, A_0 + A_1) + c_0 t\|a_0\|_{A_0} \\ &= (c_0 + 1)K(t, a_0; A_0 + A_1, A_1) + c_0 t\|a_0\|_{A_0} \end{aligned}$$

also

$$\begin{aligned} \|a_0\|_{A_0, A_0 \cap A_1} &\le (\sum_{\nu \in Z} (2^{-\nu\theta} K(2^\nu, a))^q)^{1/q}) \\ &\le (\sum_{\nu \le 0} ((C_0 + 1)2^{-\nu\theta} K(2^\nu, a))^q)^{1/q} + (\sum_{\nu \le 0} (c_0 2^{-\nu\theta} \|a_0\|_{A_0})^q)^{1/q} \end{aligned}$$

$$\leq (c_0+1)[\|a_0\|_{A_0+A_1,A_1} + \|a_0\|_{A_0}]$$

Now, the identity (1)follows.

To prove the identity (2), we note as before that one chain of inclusions is trivial. Take $a \in (A_0, A_0+A_1)_{\theta,q}$ and write $a = a_0 + a_1$ with $a_0 \in A_0, a_1 \in A_1$. Then by theorem 2.1 we have

$$K(t, a_1; A_0, A_1) \leq c_0[K(c_1t/c_0, a; A_0, A_1) + K(c_1t/c_0, a_0; A_0, A_1)]$$

$$\leq c_0[K(c_1t/c_0, a; A_0, A_1) + \|a_0\|_{A_0}]$$

for $t \geq 1, c_1/c_0 \leq 1$

$$\leq c_0[K(t, a; A_0, A_1) + \|a_0\|_{A_0}]$$

$$\leq c_0[K(t, a; A_0, A_0+A_1) + \|a_0\|_{A_0}]$$

Then

$$\|a_1\|_{A_0,A_1} \leq c_0[\|a\|_{A_0,A_0+A_1} + \|a_0\|_{A_0}]$$

And $K(t, a_1; A_0, A_1) \leq t\|a_1\|_{A_1}$ for $t \leq 1$. then $\|a_1\|A_0, A_1 \leq \|a_1\|_{A_1}$. Hence we have $a_1 \in (A_0, A_1)_{\theta,q}$.

In the proof of the next result we will have occasion to use the so called modular law for vector subspace A, B, C of a vector space Z:

$$B \subset C \Rightarrow (A+B) \cap C = (A \cap C) + B.$$

The proof of this fact is trivial.

Lemma 2.5 *Let (A_0, A_1) be a quasi-Banach couple. Then the following identities hold.*

$$(A_0+A_1, A_1)_{\theta,q} \cap (A_0, A_0+A_1)_{\theta,q} = (A_0, A_1)_{\theta,q}. \tag{3}$$

$$(A_0 \cap A_1, A_1)_{\theta,q} + (A_0, A_0 \cap A_1)_{\theta,q} = (A_0, A_1)_{\theta,q}. \tag{4}$$

Proof.
The first identity is straightforward from theorem 2.2 and the definition of $(A_0, A_1)_{\theta,q}$. Let us prove the second. As usual, one inclusion is trivial. To establish the other inclusion, take $a \in (A_0, A_1)_{\theta,q}$ and write $a = a_0 + a_1$ with $a_0 \in A_0, a_1 \in A_1$. Then

$$a_1 = a - a_0 \in [A_0 + (A_0, A_1)_{\theta,q}] \cap A_1 = [A_0 + (A_0 \cap A_1, A_1)_{\theta,q}] \cap A_1$$

$$= A_0 \cap A_1 + (A_0 \cap A_1, A_1)_{\theta,q} = (A_0 \cap A_1, A_1)_{\theta,q}$$

where we used lemma 2.4 and the modular law. Analogously(by interchanging the role of A_0 and A_1 in lemma 2.4) one also shows $a_0 \in (A_0, A_0 \cap A_1)_{\theta,q}$. Hence the second identity is proved.

Lemma 2.6 *Let (A_0, A_1) be a quasi-Banach couple. Then the following identities hold.*

$$(A_0 + A_1, A_0)_{\theta,q} \cap (A_0 + A_1, A_1)_{\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\theta,q}. \tag{5}$$

$$(A_0, A_0 \cap A_1)_{\theta,q} + (A_1, A_0 \cap A_1)_{\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\theta,q}. \tag{6}$$

Proof.
We use (2) together with the modular law and compute:

$$\begin{aligned}
(A_0{+}A_1, A_0)_{\theta,q} \cap (A_0{+}A_1, A_1)_{\theta,q} &= [A_0{+}(A_1, A_0{\cap}A_1)_{\theta,q}] \cap [(A_0, A_0{\cap}A_1)_{\theta,q}] \\
&= \{A_0 \cap [A_1 + (A_0, A_0 \cap A_1)_{\theta,q}]\} + (A_1, A_0 \cap A_1)_{\theta,q} \\
&= (A_0 \cap A_1) + (A_0, A_0 \cap A_1)_{\theta,q}]\} + (A_1, A_0 \cap A_1)_{\theta,q} \\
&= (A_0, A_0 \cap A_1)_{\theta,q}]\} + (A_1, A_0 \cap A_1)_{\theta,q} \\
&\subset (A_0 + A_1, A_0 \cap A_1)_{\theta,q} \\
&\subset (A_0 + A_1, A_0)_{\theta,q} \cap (A_0 + A_1, A_1)_{\theta,q}.
\end{aligned}$$

Lemma 2.7 *Let (A_0, A_1) be a quasi-Banach couple. Then the following identities hold.*

$$(A_0, A_1)_{\theta,q} \cap (A_0, A_1)_{1-\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\bar{\theta},q}. \tag{7}$$

$$(A_0, A_1)_{\theta,q} + (A_0, A_1)_{1-\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\underline{\theta},q}. \tag{8}$$

Proof.
Without restriction we can assume $0 < \theta \leq 2$. Using the identity (5) we write $(A_0, A_1)_{\theta,q} \cap (A_0, A_1)_{1-\theta,q} \subset (A_0 + A_1, A_0)_{1-\theta,q} \cap (A_0 + A_1, A_1)_{1-\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{1-\theta,q}$. Also, since $\theta \leq 1 - \theta$,

$$\begin{aligned}
(A_0 + A_1, A_0 \cap A_1)_{1-\theta,q} &\subset (A_0 + A_1, A_0)_{1-\theta,q} \cap (A_0 + A_1, A_1)_{1-\theta,q} \\
&\subset (A_0, A_0 + A_1)_{\theta,q} \cap (A_0 + A_1, A_1)_{\theta,q} \\
&= (A_0, A_1)_{\theta,q}
\end{aligned}$$

by lemma 2.4. Interchanging the role of A_0 and A_1 in this inclusion gives $(A_0 + A_1, A_0 \cap A_1)_{1-\theta,q} \subset (A_1, A_0)_{\theta,q} = (A_0, A_1)_{1-\theta,q}$. Hence, identity (7) is completely proved. Let us turn to (8). On the one hand, we have $(A_0{+}A_1, A_0 \cap A_1)_{\theta,q} = (A_0, A_0 \cap A_1)_{\theta,q} + (A_1, A_0 \cap A_1)_{\theta,q} \subset (A_0, A_1)_{\theta,q} + (A_1, A_0)_{\theta,q}$. On the other hand, $(A_0, A_1)_{\theta,q} = (A_0, A_0 \cap A_1)_{\theta,q} + (A_0 \cap A_1, A_1)_{\theta,q} \subset (A_0 + A_1, A_0 \cap A_1)_{\theta,q} + (A_0 + A_1, A_0 \cap A_1)_{\theta,q} \subset (A_0 + A_1, A_0 \cap A_1)_{\theta,q}$, since $\theta \leq 1 - \theta$. The concludes the proof.

Lemma 2.8 *Let (A_0, A_1) be a quasi-Banach couple. Then the following identities hold.*

$$(A_0, A_0 \cap A_1)_{\theta,q} \cap (A_0 + A_1, A_0 \cap A_1)_{1-\theta,q} = (A_0, A_0 \cap A_1)_{\bar{\theta},q}. \tag{9}$$

$$(A_0 + A_1, A_1)_{1-\theta,q} + (A_0 + A_1, A_0 \cap A_1)_{1-\theta,q} = (A_0 + A_1, A_1)_{\underline{\theta},q}. \tag{10}$$

Proof.
Let us prove (9). Using (6) and the modular law we write

$$\begin{aligned}
&(A_0, A_0 \cap A_1)_{\theta,q} \cap (A_0 + A_1, A_0 \cap A_1)_{1-\theta,q} \\
&\quad = (A_0, A_0 \cap A_1)_{\theta,q} \cap [(A_0, A_0 \cap A_1)_{1-\theta,q} + (A_1, A_0 \cap A_1)_{1-\theta,q}] \\
&\quad = (A_0, A_0 \cap A_1)_{\theta,q} \cap [(A_0, A_0 \cap A_1)_{1-\theta,q} + (A_0 \cap (A_1, A_0 \cap A_1)_{1-\theta,q})] \\
&\quad = (A_0, A_0 \cap A_1)_{\theta,q} \cap [(A_0, A_0 \cap A_1)_{1-\theta,q} + (A_0 \cap A_1)] \\
&\quad = (A_0, A_0 \cap A_1)_{\theta,q} \cap (A_0, A_0 \cap A_1)_{1-\theta,q} \\
&\quad = (A_0, A_0 \cap A_1)_{\bar{\theta},q}
\end{aligned}$$

To prove (10) we use (5) and again the modular law (but in reverse direction):

$$\begin{aligned}
&(A_0 + A_1, A_1)_{1-\theta,q} + (A_0 + A_1, A_0 \cap A_1)_{1-\theta,q} \\
&\quad = (A_0 + A_1, A_1)_{1-\theta,q} + [(A_0 + A_1, A_0)_{\theta,q} + (A_0 + A_1, A_1)_{\theta,q}] \\
&\quad = (A_0 + A_1, A_1)_{1-\theta,q} + [(A_1 + (A_0 + A_1, A_0)_{\theta,q} + \cap(A_0 + A_1, A_1)_{\theta,q}] \\
&\quad = (A_0 + A_1, A_1)_{1-\theta,q} + [(A_0 + A_1) \cap (A_0 + A_1, A_1)_{\theta,q}] \\
&\quad = (A_0 + A_1, A_1)_{1-\theta,q} + (A_0 + A_1, A_1)_{\theta,q} \\
&\quad = (A_0 + A_1, A_1)_{\underline{\theta},q}.
\end{aligned}$$

Lemma 2.9 *Let (A_0, A_1) be a quasi-Banach couple. Then the following identities hold.*

$$(A_0, A_0 \cap A_1)_{\underline{\theta},q} + (A_0 + A_1, A_0 \cap A_1)_{\bar{\theta},q} = (A_0, A_1)_{\underline{\theta},q}. \tag{11}$$

$$(A_0 + A_1, A_0)_{\bar{\theta},q} \cap (A_0 + A_1, A_0 \cap A_1)_{\underline{\theta},q} = (A_0, A_1)_{\underline{\theta},q}. \tag{12}$$

Proof.
Without restriction we can assume $0 < \theta \leq 1/2$. Then

$$\begin{aligned}
&(A_0, A_0 \cap A_1)_{\theta,q} + (A_0 + A_1, A_0 \cap A_1)_{1-\theta,q} \\
&\quad = (A_0, A_0 \cap A_1)_{\theta,q} + (A_0, A_0 \cap A_1)_{1-\theta,q} + (A_1, A_0 \cap A_1)_{1-\theta,q} \\
&\quad = (A_0, A_0 \cap A_1)_{\theta,q} + (A_1, A_0 \cap A_1)_{1-\theta,q}
\end{aligned}$$

$$= (A_0, A_1)_{\theta,q}$$

by (6) and (4). Similarly, using (5) and (3) we obtain

$$\begin{aligned}
&(A_0 + A_1, A_0)_{1-\theta,q} \cap (A_0 + A_1, A_0 \cap A_1)_{\theta,q} \\
&\qquad = (A_0 + A_1, A_0)_{1-\theta,q} \cap (A_0 + A_1, A_0)_{\theta,q} \cap (A_0 + A_1, A_1)_{\theta,q} \\
&\qquad = (A_0 + A_1, A_0)_{1-\theta,q} \cap (A_0 + A_1, A_1)_{\theta,q} \\
&\qquad = (A_0, A_1)_{\theta,q}.
\end{aligned}$$

Corollary 2.10 *Let (A_0, A_1) be a quasi-Banach couple. Then*

$$(A_0 + A_1, A_0 \cap A_1)_{\bar{\theta},q} \subset (A_0, A_1)_{\theta,q} \subset (A_0 + A_1, A_0 \cap A_1)_{\underline{\theta},q}$$

$$(A_0, A_1)_{1/2,q} = (A_0 + A_1, A_0 \cap A_1)_{1/2,q}.$$

Proof.
The first assertion is immediate from lemma 2.4. The other assertion follow from the first.

Theorem 2.11 *Let (A_0, A_1) be a quasi-Banach couple. Then the identities*

$$((A_0, A_0 \cap A_1)_{\theta,q}, (A_0 + A_1, A_1)_{\theta,q})_{\theta,q} = (A_0, A_1)_{\theta,q}$$

$$((A_0, A_0 \cap A_1)_{\theta,q}, (A_0 + A_1, A_1)_{\theta,q})_{1-\theta,q} = (A_0 + A_1, A_0 \cap A_1)_{\theta,q}.$$

hold.

Proof.
Since $(A_0, A_0 \cap A_1)_{\theta,q} \subset A_0$ and $(A_0 + A_1, A_1)_{\theta,q} \subset A_0 + A_1$, we have

$$\begin{aligned}
((A_0, A_0 \cap A_1)_{\theta,q}, (A_0 + A_1, A_1)_{\theta,q})_{\theta,q} &\subset (A_0, A_0 + A_1)_{\theta,q} \cap (A_0 + A_1, A_1)_{\theta,q} \\
&= (A_0, A_1)_{\theta,q} \\
&= (A_0, A_0 \cap A_1)_{\theta,q} + (A_0 \cap A_1, A_1)_{\theta,q} \\
&\subset ((A_0, A_0 \cap A_1)_{\theta,q}, (A_0 + A_1, A_1)_{\theta,q})_{\theta,q}.
\end{aligned}$$

Corollary 2.12 *Let (A_0, A_1) be a quasi-Banach couple with $\theta \leq 1/2$. Then*

$$((A_0 + A_1, A_0 \cap A_1)_{\theta,q}, (A_0, A_0 \cap A_1)_{\theta,q})_{1-2\theta/1-\theta,q} = (A_0, A_1)_{\theta,q}.$$

Proof.
Apply the Reiteration Theorem (see,e.g., [2,Thm. 3.5.3]) and Theorem 2.11.

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