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Convolution of Quasi-Measures

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Abstract

Let X be a compact Hausdorff space and let $\mathcal{A}$ denote the subsets of X which are either open or closed. If μ is quasi-measure on X and λ is a quasi-measure on Y, we may define a product quasi-measure $\mu \times_l \lambda$ on $X \times Y$ if either λ is a measure or μ is a $\{0, 1\}-$quasi-measure. Suppose X is CA (compact ablian)group, we define$\mu *_l \lambda$by$(\mu *_l \lambda)(E) = (\mu \times_l \lambda)(E_2)$, where E is an open set in X and $E_2 = \{(x, y) \in X^2 \ : \ x + y \in E\}$ The set function $\mu *_l \lambda$ so defined is called the convolution of μ and λ.

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1 Introduction

All spaces under consideration are compact Hausdorff. Let X be a compact Hausdorff space and let $\mathcal{A}$ denote the subsets of X which are either open or closed. A function μ is called quasi-measure if the following hold:
i) $\mu(A) \geq 0$ for all A.
ii) $A \subset B$ implies that $\mu(A) \leq \mu(B)$.
iii) $A \cap B = \emptyset$ implies $\mu(A \cup B) = \mu(A) + \mu(B)$.
iV) If U is open, $\mu(U) = sup\{\mu(K) \ : \ K \subseteq U, K \ is\ closed\}$.

V) $\mu(X) = 1$.

The primary difference between a quasi-measure and a finitely additive measure is that quasi-measure do not have to be subadditive. It is important to emphasize that a quasi-measure dose not necessarily extend to a Borel measure. Indeed, as Wheeler shows in [6], a quasi-measure extends to a Borel measure if and only if it is subadditive. The first example of quasi-measure that is not subadditive was given by Aarnes in [3]
If μ is a quasi-measure on X and ν is a quasi-measure on Y, we may define a product of quasi-measure $\mu *_l \nu$ on $X \times Y$ if either ν is measure or μ is a $\{0, 1\}-$ quasi-measure. If μ and ν are $\{0, 1\}-$quasi-measures, so is $\mu *_l \nu$. For A either open or closed in $X \times Y$, we have

$$\mu \times_l \lambda(A) = \int_Y \mu(A^y) d\lambda(y)$$

where $A^y = \{x \ : \ (x, y) \in A\}$ see in [4].

A complex function γ on a LCA (locally compact ablian) group X is called a character of X if $\mid \gamma(x) \mid = 1$ for all $x \in X$ and if the functional equation

$$\gamma(x, y) = \gamma(x)\gamma(y) \quad (x, y \in X)$$

is satisfied, The set of all continuous characters of X form a group Γ, which is a dual group of X, in case addition is defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \quad (x \in X; \gamma_1, \gamma_2 \in \Gamma).$$

Throughout this paper, the letter Γ will denote the dual group of the LCA group X.

2 Preliminary Notes

Given a quasi-measures μ, we may define a new set function $\mid \mu \mid$ on open sets by

$$\mid \mu \mid (U) = sup\{\sum \mid \mu(U_n) \mid : \ U_n \subset U \ are \ disjoint \ open \ sets\}$$

We put $\|\mu\| =\mid \mu \mid (X)$, and define M(X) to be the set of all real valued regular quasi- measures on X for which $\|\mu\|$ is finite here.
It is important to note that $\mid \mu \mid$ need not yield a quasi- measure. In particular, it is impossible to define $\mid \mu \mid$ on closed sets so that part(iii) of the definition of quasi-measure holds.

Example 2.1 *Let $X = [0,1] \times [0,1]$. In [1] and [2], it is shown how to construct the so called three point quasi-measures. This is done as follows. A subset A of X is said to be solid if both A and $X - A$ are connected. If $C = \{x_1, x_2, x_3\}$ is a set with three element, we define μ_C on solid sets by*

$$\mu_C(A) = \begin{cases} 0 & \text{if card } (A, C) \leq 1 \\ 1 & \text{if card } (A, C) \geq 2. \end{cases}$$

then there is a unique extension of μ_C to a $\{0,1\}-$valued quasi- measure on X. Now let $x_1 = (0,0)$, $x_2 = (1,0)$, $x_3 = (1,1)$, and $x_4 = (0,1)$. Let $C = \{x_1, x_2, x_3\}$, $D = \{x_2, x_3, x_4\}$. and $\mu = \mu_C - \mu_D$. If We let $U_1 = [0,1] \times [0,1/2)$ and $U_2 = [0,1] \times (1/2,1]$. We see that $2 = |\mu(U_1)| + |\mu(U_2)| \leq |\mu|(X) = \|\mu\| \leq 2$. Hence, $\mid \mu \mid (X) = 2$.Here we show that $\mid \mu \mid$ can not be extended to closed sets in a way that, it is a positive quasi- measure. In particular, the condition (iii)does not hold in the definition of quasi- measure.
Assuming such an extension exists, and $V_1 = [0,3/4) \times [0,1]$ and $V_2 = (3/4,1] \times [0,1]$. Write $K_1 = X - V_1$ and $K_2 = X - V_2$. Since $\mu_C(V_1) = \mu_D(V_1) = 0$, we see that $\mid \mu \mid (V_1) = 0 = \mid \mu \mid (V_1 \cap V_2)$. Hence $\mid \mu \mid (K_1) = 0 = \mid \mu \mid (K_1 \bigcup K_2) = 2$. Since K_1 and K_2 are disjoint, we that $\mid \mu \mid (K_2) = 0$, in other words that $\mid \mu \mid (V_2) = 2$. How ever this is not the case. In fact, $\mid \mu \mid (V_2) = 0$.

Our main reference to the theory of convolution of measure is [5]. Suppose X is a CA (compact ablian) group, and μ, λ are quasi- measures and $\{0,1\}-$ quasi- measures, respectively of $M(X)$. Let $\mu \times_l \lambda$ be their product quasi-measure on the product space $X^2 = X \times X$, and associate with each open set E in X the set

$$E_2 = \{(x,y) \in X^2 \ : \ x + y \in E\}$$

Then E_2 is an open set in X^2, since if we put $E^* = E \times X$, Then E^* is open set in $X \times X$, and since the homeomorphism of $X \times X$ onto itself which carries (x,y) to $(x-y,y)$ maps E^* onto E_2, E_2 is also an open set.
We define $\mu *_l \lambda$ by

$$(\mu *_l \lambda)(E) = (\mu \times_l \lambda)(E_2)$$

The set function $\mu *_l \lambda$ so defined is called the convolution of μ and λ.

3 Main Results

Theorem 3.1 *(a) If $\mu \in M(X)$ and $\lambda \in M(X)$, then $\mu *_l \lambda \in M(X)$.*
b) convolution is commutative and if μ or λ are $\{0,1\}-$quasi-measures then convolution is associative.
c) If μ or λ are $\{0,1\}-$quasi-measures then $\| \mu *_l \lambda \| \leq \| \mu \| . \| \lambda \|$.

Proof.
a) Since $\mu \times_l \lambda$ is a quasi- measure on X^2, it is clear that $\mu *_l \lambda$ is a quasi-measure.
If E is open set in X and if $\epsilon > 0$, the regularity of $\mu \times_l \lambda$ shows that there is a compact set $K \subset E_2$ such that

$$(\mu \times_l \lambda)(K) > (\mu \times_l \lambda)(E_2) - \epsilon$$

If C is the image of K under the map $(x, y) \longrightarrow x + y$, then C is a compact subset of E, $K \subset C_2$, and hence

$$(\mu *_l \lambda)(C) = (\mu \times_l \lambda)(C_2) \geq (\mu \times_l \lambda)(K) > (\mu *_l \lambda)(E) - \epsilon$$

b) Since X is commutative, the condition $x+y \in E$ is the same as the condition $y + x \in E$, and hence $\mu *_l \lambda = \lambda *_l \mu$.
The simplest way to prove associativity is to extend the definition of covolution to the case of n $\{0, 1\}-$quasi-measures $\mu_1, ..., \mu_n \in M(X)$ with each open set E in X associated the set

$$E_n = \{(x_1, ..., x_n) \in X^n \ : \ x_1 + ... + x_n \in E\}$$

and put

$$(\mu_1 *_l \mu_2 *_l ... *_l \mu_n)(E) = (\mu_1 \times_l \mu_2 \times_l ... \times_l \mu_n)(E_n)$$

where the measure on the right is the ordinary product quasi-measure on the product space X^n. Associativity now follows from Fubini's theorem.
c) Let χ_E be the charactristic function of the open set E in X. The definition of $(\mu *_l \lambda)(E)$ is equivalent to the equation

$$\int_X \chi_E d(\mu *_l \lambda) = \int_X \int_X \chi_E(x + y) d\mu(x) d\lambda(y)$$

Hence if f is a simple function, we have

$$\int_X f d(\mu *_l \lambda) = \int_X \int_X f(x + y) d\mu(x) \lambda(y) \tag{1}$$

and since every bounded Borl function is the uniform limit of a sequence of simple function, equation (1)holds for every bounded Borl function f.
If $| f(x) | \leq 1$ for all $x \in X$, then $| \int_X f(x + y) d\mu(x) | \leq \|\mu\|$ for all $y \in X$, and hence the right hand side of epuation (1) does not exceed $\|\mu\|.\|\lambda\|$

Theorem 3.2 *For μ and ν in M(X),*

$$carrier(\mu *_l \nu) = (carrier\ \mu).(carrier\ \nu).$$

proof.
Let A and B be the respective carriers of μ and ν. Since each is compact so is A.B, which in particular is then a Borel set. thus by the regularity of $\mu *_l \nu$, for $\varepsilon > 0$ we have an open U containing A.B for which $\mu *_l \nu(U) \leq \mu *_l \nu(A.B) + \varepsilon$.

$$1 = \mu(A).\nu(B) = \int\int \chi_A(x)\chi_B(y)d\mu(x)d\nu(y) \leq \int d(\mu *_l \nu)(x)$$

$$\leq \mu *_l \nu(U) \leq \mu *_l \nu(A.B) + \varepsilon \leq 1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\mu *_l \nu(A.B) = 1$. Moreover if U is now an open set with $(A.B) \cap U \neq \emptyset$ then we can find open sets V and W for which $V \cap A \neq \emptyset, W \cap B \neq \emptyset$, and $\overline{V}.\overline{W} \subset U$; this yields $\mu(\overline{V}).\nu(\overline{W}) \leq \mu *_l \nu(U)$, and this combines with $\mu(\overline{V}) \leq \mu(V) > 0, \nu(\overline{W}) \leq \nu(W) > 0$ to show that $\mu *_l \nu(U) > 0$. Hence A.B is indeed the carrier of $\mu *_l \nu(U)$.

4 Fourier-Stieltjes transforms.

The pairing between of X and Γ we be indicated by $(.,.)$.
If $\mu \in M(X)$, the function $\hat{\mu}$ defined on Γ by

$$\hat{\mu}(\gamma) = \int_X (-x, \gamma) d\mu(x) \quad (\gamma \in \Gamma)$$

is called the Fourier-Stieltjes transforms of μ. The set of all such functions $\hat{\mu}$ are denoted by $B(\Gamma)$.

Theorem 4.1 *a) Each $\hat{\mu} \in B(\Gamma)$ is bounded.*
b) If μ or λ $\{0,1\}-$quasi-measures, then the map $\mu \longrightarrow \hat{\mu}(\gamma)$ is, for each $\gamma \in \Gamma$, a complex homomorphism of M(G). And if $\sigma = \mu *_l \lambda$, then $\hat{\sigma} = \hat{\mu}.\hat{\lambda}$.
c) $B(\Gamma)$ is invariant under translation, under multiplication by (x, γ) for any $x \in X$ and under complex conjugation.

proof.
The definition of $\hat{\mu}$ shows immediately that $\mid \hat{\mu}(\gamma) \mid \leq \|\mu\|$ for all $\gamma \in \Gamma$.
b) Suppose $\sigma = \mu *_l \lambda$. Equation (1) in the proof of theorem 2.1 then implies that

$$\begin{aligned}\hat{\sigma}(\gamma) &= \int_X (-z, \gamma) d(\mu *_l \lambda)(z) \\ &= \int_X \int_X (-x - y, \gamma) d\mu(x) d\lambda(y) \\ &= \int_X (-x, \gamma) d\mu(x) \int_X (-y, \gamma) d\lambda(y) \\ &= \hat{\mu}(\gamma)\hat{\lambda}(\gamma)\end{aligned}$$

c) If $d\lambda(x) = (x, \gamma_0)d\mu(x)$ then $\hat{\lambda}(\gamma) = \hat{\mu}(\gamma - \gamma_0)$. If $\lambda(E) = \mu(E - x)$, then $\hat{\lambda}(\gamma) = (x, \gamma)\hat{\mu}(\gamma)$. If $\mu^*(E) = \overline{\mu(-E)}$, then the Fourier-Stieltjes transform of μ^* is the complex conjugate of $\hat{\mu}$.

For subset A of X we let $-A = \{x \in X : \quad -x \in A\}$ and denote by h_2 the homomorphism from X into itself given by

$$h_2(x) = 2x \qquad (x \in X).$$

We designate by $X^{(2)}$ and $X_{(2)}$ the image and kernel of h_2, respectively.
A quasi-measure $\mu \in M(X)$ will be termed even and atomic if for all E subset of X, $\mu(E) = \mu(-E)$ and

$$\forall A,\ B \subset X,\ A \subset B \ and \ \mu(A) > 0 \Rightarrow \mu(B) = 0.$$

The even and atomic quasi-measure belong to M(X)will be denoted by $M_{ea}(X)$.
A mapping $x \mapsto \nu_x \in M(X)(x \in X)$ will be called a homomorphism if

$$\mu_x *_l \mu_y = \mu_{x+y}$$

for any x, y$\in X$. A homomorphism $x \mapsto \nu_x \in M(X)(x \in X)$ will be said to be bounded if $sup_{x\in X}\|\mu_x\| < \infty$. A homomorphism $x \mapsto \nu_x \in M_{ea}(X)(x \in X)$ will be termed a q-homomorphism if

$$\nu_x + \nu_{-x} = \delta_x + \delta_{-x} \quad (\forall x \in X),$$

where δ_x for $x \in X$ will stand for the Dirac measure on X concentrated at x.

Let $\Phi(\Gamma)$ be the set of all odd functions φ on Γ having the following properties:
(i) $\varphi(\Gamma \setminus \Gamma_{(2)}) \subset \{-1, 1\}$ and $\varphi(\Gamma_{(2)}) = \{0\}$;
(ii) for each $x \in X$, there exists $\mu_x \in M_{ea}(X)$ such that

$$\hat{\mu_x}(\gamma) = \varphi(\gamma)(1 - (2x, -\gamma)) \quad (\gamma \in \Gamma). \tag{2}$$

Note that if $\varphi \in \Phi(\Gamma)$ and $\Gamma \neq (\Gamma_{(2)})$, then $\varphi(\Gamma \setminus \Gamma_{(2)}) = \{-1, 1\}$.

Theorem 4.2 *Let $\Phi(\Gamma)$ is nonvoid, and let φ be a function in $\Phi(\Gamma)$. Retaining the notation from the definition above, for each $x \in X$, put*

$$\nu_x = 1/2(\delta_x + \delta_{-a} + \mu_x *_l \delta_{-a}). \tag{3}$$

Then the mapping $x \mapsto \nu_x \in M_{ea}(X)(x \in X)$ is a q-homomorphism. If there exists $\mu \in M(X)$ such that $\hat{\mu} = \varphi$, then that q-homomorphism is bounded.

proof.
By equation (2) and (3), for each $x \in X$and each $\gamma \in \Gamma$,

$$\hat{\nu}_x(\gamma) = \frac{1+\varphi(\gamma)}{2}(x,\gamma) + \frac{1-\varphi(\gamma)}{2}(x,-\gamma).$$

Taking into account that φ is an 0dd function mapping $\Gamma \setminus \Gamma_{(2)}$ into $\{-1,1\}$ and $\Gamma_{(2)}$ onto $\{0\}$, we see that, for each $x \in X$ and each $\gamma \in \Gamma$,

$$\hat{\nu}(\gamma) = (x, \rho(\gamma)\gamma),$$

where

$$\rho(\gamma) = \begin{cases} \varphi(\gamma) & \text{if } \gamma \in \Gamma \setminus \Gamma_{(2)}, \\ 1 & \text{if } \gamma \in \Gamma_{(2)}. \end{cases}$$

From this it follows that, for each $x \in X$, ν_x is even and the mapping $x \mapsto \nu_x$ is a q-homomorphism of X.

If $\phi = \hat{\mu}$ for some $\mu \in M_{ea}(X)$, then, by equation (3) for each $x \in X$ we have $\mu_x = \mu - \mu *_l \delta_{2x}$, whence $\|\mu_x\| \leq 2\|\mu\|$. Thus, on account of (3), the q-homomorphism $x \mapsto \nu_x$ is bounded. The result follows.

A quasi-measure $\mu \in M(X)$ is said to be idempotent if $\mu * \mu = \mu$. The set of all idempotents in M(X) will be denoted by J(X).

If $\mu \in J(X)$, then $\hat{\mu}^2 = \hat{\mu}$, so that $\hat{\mu}(\gamma) = 1$ or 0 for all $\gamma \in \Gamma$. Now let us

$$S(\mu) = \{\gamma \in \Gamma \;:\; \hat{\mu}(\gamma) = 1\} \qquad (\mu \in J(X)).$$

The problem of finding all $\mu \in J(X)$ is obviously equivalent to the problem of finding all subsets of Γ whose characteristic function belong to $B(\Gamma)$. If μ and ν are in J(X), then so are the quasi-measures $\mu *_l \nu$ and $\mu \vee \nu = \mu + \nu - \mu *_l \nu$, as well as $\delta_0 - \nu$, where δ_0 is the point measure of unit norm concentrated at the point 0 in X, because

$$S(\mu *_l \nu) = S(\mu) \cap S(\nu)\ , \quad S(\mu \vee \nu) = S(\mu) \cup S(\nu)$$

Since $\hat{\mu}$ is continuous, $S(\mu)$ is open and closed, for every $\mu \in J(X)$. Consequently, if Γ is connected, there are only two possibilities for $S(\mu)$: $S(\mu) = \Gamma$ or $S(\mu)$ is empty. In other words, δ_0 and 0 are the only members of J(X).

Lemma 4.3 *$S(\mu)$ is an open subgroup and coset in Γ.*

proof.
Let $\mu \in J(X)$, $\mu \neq 0$ and $\mu \geq 0$. Then $\hat{\mu}(0) = 1$. If γ_0 and γ_1 are in $S(\mu)$, then $-\gamma_1 \in S(\mu)$, and

$$|\hat{\mu}(\gamma_0 - \gamma_1) - \hat{\mu}(\gamma_0)| \leq 2\hat{\mu}(0) Re[\hat{\mu}(0) - \hat{\mu}(-\gamma_1)] = 0.$$

Hence $\gamma_0 - \gamma_1 \in S(\mu)$, and we conclude that $S(\mu)$ is an open subgroup of Γ. If $\mu \in J(X)$ and $\mu \neq 0$ then $\|\mu\| = \|\mu *_l \mu\| \leq \|\mu\|^2$, so that $\|\mu\| \geq 1$. Suppose $\|\mu\| = 1$. Setting $d\nu(x) = (x, \gamma)d\hat{\mu}(x)$, proper choice of γ assures that $\hat{\nu}(0) = 1$. Then

$$1 = \hat{\nu}(0) = \nu(X) \leq \|\nu\| = 1$$

hence $\nu(X) = \|\nu\|,\ \nu \geq 0$, and the preceding result implies: If $\mu \in J(X)$ and $\|\mu\| = 1$, then $S(\mu)$ is an open coset in Γ.

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