Convolution of Quasi-Measures

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Abstract

Let $X$ be a compact Hausdorff space and let $A$ denote the subsets of $X$ which are either open or closed. If $\mu$ is quasi-measure on $X$ and $\lambda$ is a quasi-measure on $Y$, we may define a product quasi-measure $\mu \times \lambda$ on $X \times Y$ if either $\lambda$ is a measure or $\mu$ is a $\{0,1\}$-quasi-measure. Suppose $X$ is CA (compact abelian) group, we define $\mu \ast \lambda$ by $(\mu \ast \lambda)(E) = (\mu \times \lambda)(E_2)$, where $E$ is an open set in $X$ and $E_2 = \{ (x,y) \in X^2 : x + y \in E \}$. The set function $\mu \ast \lambda$ so defined is called the convolution of $\mu$ and $\lambda$.

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1 Introduction

All spaces under consideration are compact Hausdorff. Let $X$ be a compact Hausdorff space and let $A$ denote the subsets of $X$ which are either open or closed. A function $\mu$ is called quasi-measure if the following hold:
i) $\mu(A) \geq 0$ for all $A$.
ii) $A \subset B$ implies that $\mu(A) \leq \mu(B)$.
iii) $A \cap B = \emptyset$ implies $\mu(A \cup B) = \mu(A) + \mu(B)$.
iV) If $U$ is open, $\mu(U) = \sup \{ \mu(K) : K \subseteq U, K \text{ is closed} \}$. 


V) \( \mu(X) = 1 \).

The primary difference between a quasi-measure and a finitely additive measure is that quasi-measure do not have to be subadditive. It is important to emphasize that a quasi-measure does not necessarily extend to a Borel measure. Indeed, as Wheeler shows in [6], a quasi-measure extends to a Borel measure if and only if it is subadditive. The first example of quasi-measure that is not subadditive was given by Aarnes in [3].

If \( \mu \) is a quasi-measure on \( X \) and \( \nu \) is a quasi-measure on \( Y \), we may define a product of quasi-measure \( \mu \ast_1 \nu \) on \( X \times Y \) if either \( \nu \) is measure or \( \mu \) is a \( \{0, 1\} \) quasi-measure. If \( \mu \) and \( \nu \) are \( \{0, 1\} \) quasi-measures, so is \( \mu \ast_1 \nu \). For \( A \) either open or closed in \( X \times Y \), we have

\[
\mu \times_1 \lambda(A) = \int_Y \mu(A^y)d\lambda(y)
\]

where \( A^y = \{x : (x, y) \in A\} \) see in [4].

A complex function \( \gamma \) on a LCA (locally compact abelian) group \( X \) is called a character of \( X \) if \( |\gamma(x)| = 1 \) for all \( x \in X \) and if the functional equation

\[
\gamma(x, y) = \gamma(x)\gamma(y) \quad (x, y \in X)
\]

is satisfied. The set of all continuous characters of \( X \) form a group \( \Gamma \), which is a dual group of \( X \), in case addition is defined by

\[
(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \quad (x \in X; \gamma_1, \gamma_2 \in \Gamma).
\]

Throughout this paper, the letter \( \Gamma \) will denote the dual group of the LCA group \( X \).

## 2 Preliminary Notes

Given a quasi-measures \( \mu \), we may define a new set function \( |\mu| \) on open sets by

\[
|\mu|(U) = \sup\{\sum |\mu(U_n)| : U_n \subset U \text{ are disjoint open sets}\}
\]

We put \( \|\mu\| = |\mu|(X) \), and define \( M(X) \) to be the set of all real valued regular quasi-measures on \( X \) for which \( \|\mu\| \) is finite here. It is important to note that \( |\mu| \) need not yield a quasi-measure. In particular, it is impossible to define \( |\mu| \) on closed sets so that part(iii) of the definition of quasi-measure holds.
Example 2.1 Let $X = [0, 1] \times [0, 1]$. In [1] and [2], it is shown how to construct the so called three point quasi-measures. This is done as follows. A subset $A$ of $X$ is said to be solid if both $A$ and $X - A$ are connected. If $C = \{x_1, x_2, x_3\}$ is a set with three elements, we define $\mu_C$ on solid sets by

$$
\mu_C(A) = \begin{cases} 
0 & \text{if card}(A, C) \leq 1 \\
1 & \text{if card}(A, C) \geq 2.
\end{cases}
$$

then there is a unique extension of $\mu_C$ to a $\{0, 1\}$–valued quasi-measure on $X$. Now let $x_1 = (0, 0)$, $x_2 = (1, 0)$, $x_3 = (1, 1)$, and $x_4 = (0, 1)$. Let $C = \{x_1, x_2, x_3\}$, $D = \{x_2, x_3, x_4\}$, and $\mu = \mu_C - \mu_D$. If we let $U_1 = [0, 1] \times [0, 1/2]$ and $U_2 = [0, 1] \times (1/2, 1]$. We see that $2 = |\mu(U_1)| + |\mu(U_2)| \leq |\mu|(X) = \|\mu\| \leq 2$. Hence, $|\mu|(X) = 2$. Here we show that $|\mu|$ cannot be extended to closed sets in a way that, it is a positive quasi-measure. In particular, the condition (iii) does not hold in the definition of quasi-measure.

Assuming such an extension exists, and $V_1 = [0, 3/4) \times [0, 1]$ and $V_2 = (3/4, 1) \times [0, 1]$. Write $K_1 = X - V_1$ and $K_2 = X - V_2$. Since $\mu_C(V_1) = \mu_D(V_1) = 0$, we see that $|\mu|(V_1) = 0 = |\mu|(V_1 \cap V_2)$. Hence $|\mu|(K_1) = 0 = |\mu|(K_1 \cup K_2) = 2$. Since $K_1$ and $K_2$ are disjoint, we that $|\mu|(K_2) = 0$, in other words that $|\mu|(V_2) = 2$. How ever this is not the case. In fact, $|\mu|(V_2) = 0$.

Our main reference to the theory of convolution of measure is [5]. Suppose $X$ is a CA (compact abelian) group, and $\mu, \lambda$ are quasi-measures and $\{0, 1\}$–quasi-measures, respectively of $M(X)$. Let $\mu \times_1 \lambda$ be their product quasi-measure on the product space $X^2 = X \times X$, and associate with each open set $E$ in $X$ the set

$$
E_2 = \{(x, y) \in X^2 : x + y \in E\}
$$

Then $E_2$ is an open set in $X^2$, since if we put $E^* = E \times X$, Then $E^*$ is open set in $X \times X$, and since the homeomorphism of $X \times X$ onto itself which carries $(x, y)$ to $(x - y, y)$ maps $E^*$ onto $E_2$. $E_2$ is also an open set.

We define $\mu \times_1 \lambda$ by

$$
(\mu \times_1 \lambda)(E) = (\mu \times_1 \lambda)(E_2)
$$

The set function $\mu \times_1 \lambda$ so defined is called the convolution of $\mu$ and $\lambda$.

3 Main Results

Theorem 3.1 (a) If $\mu \in M(X)$ and $\lambda \in M(X)$, then $\mu \times_1 \lambda \in M(X)$.

b) convolution is commutative and if $\mu$ or $\lambda$ are $\{0, 1\}$–quasi-measures then convolution is associative.

c) If $\mu$ or $\lambda$ are $\{0, 1\}$–quasi-measures then $\| \mu \times_1 \lambda \| \leq \| \mu \| \cdot \| \lambda \|$. 
Proof.
a) Since $\mu \times_1 \lambda$ is a quasi-measure on $X^2$, it is clear that $\mu \ast_1 \lambda$ is a quasi-measure.
If $E$ is open set in $X$ and if $\epsilon > 0$, the regularity of $\mu \times_1 \lambda$ shows that there is a compact set $K \subset E_2$ such that
\[(\mu \times_1 \lambda)(K) > (\mu \times_1 \lambda)(E_2) - \epsilon\]
If $C$ is the image of $K$ under the map $(x, y) \mapsto x + y$, then $C$ is a compact subset of $E$, $K \subset C_2$, and hence
\[(\mu \ast_1 \lambda)(C) = (\mu \times_1 \lambda)(C_2) \geq (\mu \times_1 \lambda)(K) > (\mu \ast_1 \lambda)(E) - \epsilon\]

b) Since $X$ is commutative, the condition $x + y \in E$ is the same as the condition $y + x \in E$, and hence $\mu \ast_1 \lambda = \lambda \ast_1 \mu$.

The simplest way to prove associativity is to extend the definition of convolution to the case of $n \{0, 1\}$-quasi-measures $\mu_1, ..., \mu_n \in M(X)$ with each open set $E$ in $X$ associated the set
\[E_n = \{(x_1, ..., x_n) \in X^n : x_1 + ... + x_n \in E\}\]
and put
\[(\mu_1 \ast_1 \mu_2 \ast_1 ... \ast_1 \mu_n)(E) = (\mu_1 \times_1 \mu_2 \times_1 ... \times_1 \mu_n)(E_n)\]
where the measure on the right is the ordinary product quasi-measure on the product space $X^n$. Associativity now follows from Fubini's theorem.

c) Let $\chi_E$ be the characteristic function of the open set $E$ in $X$. The definition of $(\mu \ast_1 \lambda)(E)$ is equivalent to the equation
\[\int_X \chi_E d(\mu \ast_1 \lambda) = \int_X \int_X \chi_E(x + y) d\mu(x) d\lambda(y)\]
Hence if $f$ is a simple function, we have
\[\int_X f d(\mu \ast_1 \lambda) = \int_X \int_X f(x + y) d\mu(x) d\lambda(y)\] (1)
and since every bounded Borel function is the uniform limit of a sequence of simple function, equation (1) holds for every bounded Borel function $f$.

If $|f(x)| \leq 1$ for all $x \in X$, then $|\int_X f(x + y) d\mu(x)| \leq \|\mu\|$ for all $y \in X$, and hence the right hand side of equation (1) does not exceed $\|\mu\| \cdot \|\lambda\|$.

**Theorem 3.2** For $\mu$ and $\nu$ in $M(X)$,
\[\text{carrier}(\mu \ast_1 \nu) = (\text{carrier } \mu) \cdot (\text{carrier } \nu).\]
proof.
Let $A$ and $B$ be the respective carriers of $\mu$ and $\nu$. Since each is compact so is $A.B$, which in particular is then a Borel set. Thus by the regularity of $\mu *_1 \nu$, for $\varepsilon > 0$ we have an open $U$ containing $A.B$ for which $\mu *_1 \nu(U) \leq \mu *_1 \nu(A.B) + \varepsilon$.

$$1 = \mu(A).\nu(B) = \int \int \chi_A(x)\chi_B(y)d\mu(x)d\nu(y) \leq \int d(\mu *_1 \nu)(x)$$

$$\leq \mu *_1 \nu(U) \leq \mu *_1 \nu(A.B) + \varepsilon \leq 1 + \varepsilon.$$  

Since $\varepsilon > 0$ is arbitrary, $\mu *_1 \nu(A.B) = 1$. Moreover if $U$ is now an open set with $(A.B) \cap U \neq \emptyset$ then we can find open sets $V$ and $W$ for which $V \cap A \neq \emptyset, W \cap B \neq \emptyset$, and $V.W \subset U$; this yields $\mu(V).\nu(W) \leq \mu *_1 \nu(U)$, and this combines with $\mu(V) \leq \mu(V) > 0, \nu(W) \leq \nu(W) > 0$ to show that $\mu *_1 \nu(U) > 0$. Hence $A.B$ is indeed the carrier of $\mu *_1 \nu(U)$.

4 Fourier-Stieltjes transforms.

The pairing between of $X$ and $\Gamma$ we be indicated by $(\cdot., \cdot.)$. If $\mu \in M(X)$, the function $\hat{\mu}$ defined on $\Gamma$ by

$$\hat{\mu}(\gamma) = \int_X (-x, \gamma)d\mu(x) \quad (\gamma \in \Gamma)$$

is called the Fourier-Stieltjes transforms of $\mu$. The set of all such functions $\hat{\mu}$ are denoted by $B(\Gamma)$.

**Theorem 4.1**  

a) Each $\hat{\mu} \in B(\Gamma)$ is bounded.  
b) If $\mu$ or $\lambda \{0, 1\}$–quasi-measures, then the map $\mu \rightarrow \hat{\mu}(\gamma)$ is, for each $\gamma \in \Gamma$, a complex homomorphism of $M(G)$. And if $\sigma = \mu *_1 \lambda$, then $\hat{\sigma} = \hat{\mu} \hat{\lambda}$.  
c) $B(\Gamma)$ is invariant under translation, under multiplication by $(x, \gamma)$ for any $x \in X$ and under complex conjugation.

**proof.**

The definition of $\hat{\mu}$ shows immediately that $|\hat{\mu}(\gamma)| \leq \|\mu\|$ for all $\gamma \in \Gamma$.  
b) Suppose $\sigma = \mu *_1 \lambda$. Equation (1) in the proof of theorem 2.1 then implies that

$$\hat{\sigma}(\gamma) = \int_X (-z, \gamma)d(\mu *_1 \lambda)(z)$$

$$= \int_X \int_X (-x - y, \gamma)d\mu(x)d\lambda(y)$$

$$= \int_X (-x, \gamma)d\mu(x) \int_X (-y, \gamma)d\lambda(y)$$

$$= \hat{\mu}(\gamma)\hat{\lambda}(\gamma)$$
c) If \( d\lambda(x) = (x, \gamma_0) d\mu(x) \) then \( \hat{\lambda}(\gamma) = \hat{\mu}(\gamma - \gamma_0) \). If \( \lambda(E) = \mu(E - x) \), then \( \lambda(\gamma) = (x, \gamma) \hat{\mu}(\gamma) \). If \( \mu^*(E) = \overline{\mu(-E)} \), then the Fourier-Stieltjes transform of \( \mu^* \) is the complex conjugate of \( \hat{\mu} \).

For subset \( A \) of \( X \) we let \( -A = \{ x \in X : -x \in A \} \) and denote by \( h_2 \) the homomorphism from \( X \) into itself given by

\[
h_2(x) = 2x \quad (x \in X).
\]

We designate by \( X^{(2)} \) and \( X_{(2)} \) the image and kernel of \( h_2 \), respectively. A quasi-measure \( \mu \in M(X) \) will be termed even and atomic if for all \( E \) subset of \( X \), \( \mu(E) = \mu(-E) \) and

\[
\forall A, B \subset X, A \subset B \text{ and } \mu(A) > 0 \Rightarrow \mu(B) = 0.
\]

The even and atomic quasi-measure belong to \( M(X) \) will be denoted by \( M_{ea}(X) \).

A mapping \( x \mapsto \nu_x \in M(X)(x \in X) \) will be called a homomorphism if

\[
\mu_x * \nu_y = \mu_{x+y}
\]

for any \( x, y \in X \). A homomorphism \( x \mapsto \nu_x \in M(X)(x \in X) \) will be said to be bounded if \( \sup_{x \in X} \| \mu_x \| < \infty \). A homomorphism \( x \mapsto \nu_x \in M_{ea}(X)(x \in X) \) will be termed a q-homomorphism if

\[
\nu_x + \nu_{-x} = \delta_x + \delta_{-x} \quad (\forall x \in X),
\]

where \( \delta_x \) for \( x \in X \) will stand for the Dirac measure on \( X \) concentrated at \( x \).

Let \( \Phi(\Gamma) \) be the set of all odd functions \( \varphi \) on \( \Gamma \) having the following properties:

(i) \( \varphi(\Gamma \setminus \Gamma_{(2)}) \subset \{-1, 1\} \) and \( \varphi(\Gamma_{(2)}) = \{0\} \);

(ii) for each \( x \in X \), there exists \( \mu_x \in M_{ea}(X) \) such that

\[
\hat{\mu}_x(\gamma) = \varphi(\gamma)(1 - (2x, -\gamma)) \quad (\gamma \in \Gamma).
\]

Note that if \( \varphi \in \Phi(\Gamma) \) and \( \Gamma \neq \Gamma_{(2)} \), then \( \varphi(\Gamma \setminus \Gamma_{(2)}) = \{-1, 1\} \).

**Theorem 4.2** Let \( \Phi(\Gamma) \) is nonvoid, and let \( \varphi \) be a function in \( \Phi(\Gamma) \). Retaining the notation from the definition above, for each \( x \in X \), put

\[
\nu_x = 1/2(\delta_x + \delta_{-x} + \mu_x * \delta_{-a}).
\]

Then the mapping \( x \mapsto \nu_x \in M_{ea}(X)(x \in X) \) is a q-homomorphism. If there exists \( \mu \in M(X) \) such that \( \hat{\mu} = \varphi \), then that q-homomorphism is bounded.
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**proof.**

By equation (2) and (3), for each \( x \in X \) and each \( \gamma \in \Gamma \),

\[
\hat{\nu}_x(\gamma) = \frac{1 + \varphi(\gamma)}{2} (x, \gamma) + \frac{1 - \varphi(\gamma)}{2} (x, -\gamma).
\]

Taking into account that \( \varphi \) is an odd function mapping \( \Gamma \setminus \Gamma(2) \) into \( \{-1, 1\} \) and \( \Gamma(2) \) onto \( \{0\} \), we see that, for each \( x \in X \) and each \( \gamma \in \Gamma \),

\[
\hat{\nu}(\gamma) = (x, \rho(\gamma))
\]

where

\[
\rho(\gamma) = \begin{cases} 
\varphi(\gamma) & \text{if } \gamma \in \Gamma \setminus \Gamma(2), \\
1 & \text{if } \gamma \in \Gamma(2).
\end{cases}
\]

From this it follows that, for each \( x \in X \), \( \nu_x \) is even and the mapping \( x \mapsto \nu_x \) is a q-homomorphism of \( X \).

If \( \phi = \hat{\mu} \) for some \( \mu \in M_{eq}(X) \), then, by equation (3) for each \( x \in X \) we have \( \mu_x = \mu - \mu \ast l \delta_{2x} \), whence \( \|\mu_x\| \leq 2\|\mu\| \). Thus, on account of (3), the q-homomorphism \( x \mapsto \nu_x \) is bounded. The result follows.

A quasi-measure \( \mu \in M(X) \) is said to be idempotent if \( \mu \ast \mu = \mu \). The set of all idempotents in \( M(X) \) will be denoted by \( J(X) \).

If \( \mu \in J(X) \), then \( \hat{\mu}^2 = \hat{\mu} \), so that \( \hat{\mu}(\gamma) = 1 \) or 0 for all \( \gamma \in \Gamma \). Now let us

\[
S(\mu) = \{ \gamma \in \Gamma : \hat{\mu}(\gamma) = 1 \} \quad (\mu \in J(X)).
\]

The problem of finding all \( \mu \in J(X) \) is obviously equivalent to the problem of finding all subsets of \( \Gamma \) whose characteristic function belong to \( B(\Gamma) \). If \( \mu \) and \( \nu \) are in \( J(X) \), then so are the quasi-measures \( \mu \ast_1 \nu \) and \( \mu \lor \nu = \mu + \nu - \mu \ast_1 \nu \), as well as \( \delta_0 - \nu \), where \( \delta_0 \) is the point measure of unit norm concentrated at the point 0 in \( X \), because

\[
S(\mu \ast_1 \nu) = S(\mu) \cap S(\nu), \quad S(\mu \lor \nu) = S(\mu) \cup S(\nu)
\]

Since \( \hat{\mu} \) is continuous, \( S(\mu) \) is open and closed, for every \( \mu \in J(X) \). Consequently, if \( \Gamma \) is connected, there are only two possibilities for \( S(\mu) : S(\mu) = \Gamma \) or \( S(\mu) \) is empty. In other words, \( \delta_0 \) and 0 are the only members of \( J(X) \).

**Lemma 4.3** \( S(\mu) \) is an open subgroup and coset in \( \Gamma \).

**proof.**

Let \( \mu \in J(X) \), \( \mu \neq 0 \) and \( \mu \geq 0 \). Then \( \hat{\mu}(0) = 1 \). If \( \gamma_0 \) and \( \gamma_1 \) are in \( S(\mu) \), then \( -\gamma_1 \in S(\mu) \), and

\[
|\hat{\mu}(\gamma_0 - \gamma_1) - \hat{\mu}(\gamma_0)| \leq 2\hat{\mu}(0) Re[\hat{\mu}(0) - \hat{\mu}(-\gamma_1)] = 0.
\]
Hence $\gamma_0 - \gamma_1 \in S(\mu)$, and we conclude that $S(\mu)$ is an open subgroup of $\Gamma$. If $\mu \in J(X)$ and $\mu \neq 0$ then $\|\mu\| = \|\mu * l_\mu\| \leq \|\mu\|^2$, so that $\|\mu\| \geq 1$. Suppose $\|\mu\| = 1$. Setting $d\nu(x) = (x, \gamma)d\tilde{\mu}(x)$, proper choice of $\gamma$ assures that $\tilde{\nu}(0) = 1$. Then
\[
1 = \tilde{\nu}(0) = \nu(X) \leq \|\nu\| = 1
\]
hence $\nu(X) = \|\nu\|$, $\nu \geq 0$, and the preceding result implies: If $\mu \in J(X)$ and $\|\mu\| = 1$, then $S(\mu)$ is an open coset in $\Gamma$.

References


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