A Quadrature Formula for the Approximation of Cauchy Type Singular Integrals on the Interval

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Abstract

The paper deals with the construction of an efficient quadrature formula for singular integrals (SI) of Cauchy type. The construction of quadrature formula is based on modification of discrete vortex method and interpolation linear spline. The estimations of errors are obtained in the classes of $H^\alpha([-1,1],K)$ and $C^1([-1,1])$. Numerical analysis are also given.

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1 Introduction

This paper is concerned with Cauchy type singular integrals of the form

$$\int_{-1}^{1} \frac{f(t)}{t-x} dt, \quad x \in (-1,1)$$  \hspace{1cm} (1)

where $f(t) \in H^\alpha(K,[-1,1])$.

Numerical evaluation of (1) has been considered in [2], where the convergence of quadrature formula was provided in the interval $x \in [t_j + \delta, t_{j+1} - \delta]$, where $\{t_j\}$ form an equal partition of $[-1,1]$ and $\delta \in [h/4,3h/4]$, $h = \frac{2}{N+1}$, $t_{j+1} = t_j + h$. The paper discusses the convergence of quadrature formula for any singular point $x$ over the closed interval $[t_j, t_{j+1}]$, and the rate of convergence is identified in the classes of functions $H^\alpha([-1,1],K)$ and
\[ C^1([-1, 1]). \] Numerical examples are given to verify the validity of quadrature formula. These examples are also compared with the quadrature sum obtained by discrete vortex method \[1\].

\section{Construction of the quadrature formula (QF)}

Let us examine SI (1). We divide the interval \([-1, 1]\) into \(N + 1\) equal subintervals \([t_k, t_{k+1}]\), \(k = 0, ..., N\) where \(t_k = -1 + kh, k = 0, ..., N + 1\). The set \(E = \{t_k, k = 1, ..., N\}\) is called a canonic partition of the interval \([-1, 1]\) (see \[1\]). Let \(Q(j) = \{j - 1, j, j + 1, j = 1, ..., N\}\) where it is assumed that \(\{j\}\) is a fixed integer.

We consider two cases:

1. singular point \(x\) does not coincide with the knot points \(-1 = t_0 < t_1 < t_2 < ... < t_N < t_{N+1} = 1\), that is \(x = t_j + \varepsilon, j = 1, ..., N, \) where \(\varepsilon \in (0, h)\).

2. singular point \(x\) coincides with knot points, that is \(x = t_j, j = 1, ..., N\).

Now let \(s_{1,\nu}(t)\) and \(s_{2,\nu}(t)\) be a linear interpolation spline \[3\] such that

\[ s_{1,\nu}(t) = \frac{1}{h}[(t_{\nu+1} - t)f(t_{\nu}) + (t - t_{\nu})f(t_{\nu+1})], \quad \nu = 0, ..., N, \quad t \in [t_{\nu}, t_{\nu+1}] \] (2)

\[ s_{2,\nu}(t) = \frac{1}{2h}[(t_{\nu+1} - t)f(t_{\nu-1}) + (t - t_{\nu-1})f(t_{\nu+1})], \quad \nu = 0, ..., N, \quad t \in [t_{\nu-1}, t_{\nu+1}] \] (3)

which has the following properties:

\(a\) if \(f(t) = c\) then \(s_{1,\nu}(t) = s_{2,\nu}(t) = c\)

\(b\) if \(f(t) = at + b\), then \(s_{1,\nu}(t) = s_{2,\nu}(t) = at + b\).

In the first case we use interpolation spline function \(s_{1,\nu}(t)\) (2) to construct QF for SI (1) yields

\[ \frac{1}{t - (t_j + \varepsilon)} \int_{-1}^{1} f(t)dt = \sum_{k=0, k \notin Q(j)}^N \int_{t_k}^{t_{k+1}} \frac{f(t)dt}{t - (t_j + \varepsilon)} + \sum_{\nu=0}^{j+1} \int_{t_{\nu-1}}^{t_{\nu+1}} \frac{f(t)dt}{t - (t_j + \varepsilon)} \]

\[ = \sum_{k=0, k \notin Q(j)}^{N+1} A_k(t_j + \varepsilon)f(t_k) + \sum_{\nu=j+1}^{j+1} \int_{t_{\nu-1}}^{t_{\nu+1}} \frac{s_{1,\nu}(t)dt}{t - (t_j + \varepsilon)} + R(t_j + \varepsilon), \] (4)

where

\[ A_k(t_j + \varepsilon) = \frac{h}{t_k - (t_j + \varepsilon)}, k = 1, ..., j - 2, j + 3, ..., N. \] (5)

The coefficients \(A_k\) are similar to the coefficients obtained by discrete vortex method \[1\]. Since the quadrature formula (4) is exact for the linear function
f(t) and by the properties of linear interpolation spline function, the coefficients $A_0$ and $A_{N+1}$ are found as

\[
A_0(t_j + \varepsilon) = \frac{1}{2} \left[ \int_{t_j}^{t_{j+2}} \frac{(1-t)dt}{t-(t_j + \varepsilon)} - \sum_{k=1, k \not\in Q(j)+2}^{N} \frac{(1-t_k)h}{t_k - (t_j + \varepsilon)} \right],
\]

\[
A_{N+1}(t_j + \varepsilon) = \frac{1}{2} \left[ \int_{t_{j-1}}^{t_j} \frac{(1+t)dt}{t-(t_j + \varepsilon)} - \sum_{k=1, k \not\in Q(j)+2}^{N} \frac{(1+t_k)h}{t_k - (t_j + \varepsilon)} \right].
\]

Substituting (5) and (6) into (4), we obtain

\[
\int_{-1}^{1} \frac{\varphi(t)dt}{t-(t_j + \varepsilon)} = \sum_{k=1, k \not\in j+2, k \not\in Q(j)}^{N} \frac{\varphi(t_k)h}{t_k - (t_j + \varepsilon)} + \sum_{\nu= j-1}^{j+1} \int_{t_{\nu}}^{t_{\nu+1}} \frac{s_{1,\nu}(t)dt}{t-(t_j + \varepsilon)} + R_N(t_j + \varepsilon),
\]

where

\[
\varphi(t) = f(t) - \frac{1}{2} [(1-t)f(-1) + (1+t)f(1)],
\]

\[
s_{1,\nu}(t) = s_{1,\nu}(t) - \frac{1}{2} [(1-t)f(-1) + (1+t)f(1)].
\]

Evaluating corresponding integral in (7) one has the following quadrature formula for actual evaluation

\[
\int_{-1}^{1} \frac{f(t)dt}{t-(t_j + \varepsilon)} = \sum_{k=0}^{N+1} A_k(t_j + \varepsilon)f(t_k) + R_N(t_j + \varepsilon),
\]

where

\[
A_k(t_j + \varepsilon) = \frac{h}{t_k - (t_j + \varepsilon)} , \quad k = 1, ..., j - 2, j + 3, ..., N,
\]

\[
A_0(t_j + \varepsilon) = \frac{1}{2} [(1 - (t_j + \varepsilon))T_N(t_j + \varepsilon) - 2h],
\]

\[
A_{N+1}(t_j + \varepsilon) = \frac{1}{2} [(1 + (t_j + \varepsilon))T_N(t_j + \varepsilon) + 2h],
\]

\[
A_{j-1}(t_j + \varepsilon) = - \left( 1 + \frac{\varepsilon}{h \ln \frac{h}{h+\varepsilon}} \right),
\]

\[
A_j(t_j + \varepsilon) = \ln \frac{h-\varepsilon}{h+\varepsilon} + \varepsilon \ln \frac{\varepsilon^2}{h^2 - \varepsilon^2},
\]

\[
A_{j+1}(t_j + \varepsilon) = 2\ln \frac{2h-\varepsilon}{h-\varepsilon} + \varepsilon \ln \frac{(h-\varepsilon)^2}{\varepsilon(2h-\varepsilon)},
\]

\[
A_{j+2}(t_j + \varepsilon) = 1 - \frac{h}{h+\varepsilon} \ln \frac{2h-\varepsilon}{h-\varepsilon},
\]

\[
T_N(t_j + \varepsilon) = \ln \frac{1 - (t_j + \varepsilon)h + \varepsilon}{1 + (t_j + \varepsilon)2h - \varepsilon} - \sum_{k=1, k \not\in Q(j)+j+2}^{N} \frac{h}{t_k - (t_j + \varepsilon)}.
\]
For the second case, we write

$$\int_{-1}^{1} \frac{f(t)dt}{t - t_j} = \sum_{k=0, k \not\in \mathbb{Q}(j)}^{N} \int_{t_k}^{t_{k+1}} \frac{f(t)dt}{t - t_j} + \int_{t_j}^{t_{j+1}} \frac{f(t)dt}{t - t_j}$$

$$= \sum_{k=0, k \not\in \mathbb{Q}(j)}^{N+1} B_k(t_j) f(t_k) + \int_{t_j}^{t_{j+1}} s_{2,\nu}(t) dt \frac{t_{j+1} - t_j}{t - t_j} + R(t_j),$$

(11)

where

$$B_k(t_j) = \frac{h}{t_k - t_j}, k = 1, \ldots, j - 2, j + 2, \ldots, N.$$  

(12)

The interpolation linear spline function $s_{2,\nu}$ (3) is used on the second integral of the right hand side of (11). As in the first case, the coefficients $B_0$ and $B_{N+1}$ are found as

$$B_0(t_j) = \frac{1}{2} \left[ \int_{-1}^{1} \frac{(1 - t)dt}{t - t_j} - \int_{t_j}^{t_{j+1}} \frac{(1 - t)dt}{t - t_j} - \sum_{k=1, k \not\in \mathbb{Q}(j)}^{N} \frac{(1 - t_k)h}{t_k - t_j} \right]$$

$$B_{N+1}(t_j) = \frac{1}{2} \left[ \int_{-1}^{1} \frac{(1 + t)dt}{t - t_j} - \int_{t_j}^{t_{j+1}} \frac{(1 + t)dt}{t - t_j} - \sum_{k=1, k \not\in \mathbb{Q}(j)}^{N} \frac{(1 + t_k)h}{t_k - t_j} \right].$$

(13)

Substituting (12) and (13) into (11) yields

$$\int_{-1}^{1} \frac{\varphi(t)dt}{t - t_j} = \sum_{k=1, k \not\in \mathbb{Q}(j)}^{N} \varphi(t_k) h \frac{t_{j+1} - t_j}{t_k - t_j} + \int_{t_j}^{t_{j+1}} s_{2,\nu}^{*}(t) dt \frac{t_{j+1} - t_j}{t - t_j} + R(t_j)$$

(14)

where $\varphi(t)$ is defined by (8) and

$$s_{2,\nu}^{*}(t) = s_{2,\nu}(t) - \frac{1}{2} \left[ (1 - t)f(-1) + (1 + t)f(1) \right].$$

Evaluating integral on the right side of the equation (14), we obtain the quadrature formula for actual evaluation

$$\int_{-1}^{1} \frac{f(t)dt}{t - t_j} = \sum_{k=0}^{N+1} B_k(t_j) f(t_k) + R(t_j),$$

(15)

where
\[ B_k(t_j) = \frac{h}{t_k - t_j}, \quad k = 1, \ldots, j - 2, j + 2, \ldots, N, \]
\[ B_0(t_j) = \frac{1}{2} \left[ (1 - t_j)T_N(t_j) - 2h \right], \]
\[ B_{N+1}(t_j) = \frac{1}{2} \left[ (1 + t_j)T_N(t_j) + 2h \right], \]
\[ B_{j-1}(t_j) = -1, \]
\[ B_j(t_j) = 0, \]
\[ B_{j+1}(t_j) = 2 \ln 2, \]
\[ T_N(t_j) = \ln \left| \frac{1 - t_j}{2(1 + t_j)} \right| - \sum_{k=1, k \notin Q(j)}^{N} \frac{h}{t_k - t_j}. \]

3 Estimation of errors

We give some definitions of classes of functions, which will be used in the sequel.

1. \( H^\alpha([-1, 1], K) \) - is a class of function \( f(x) \), which is satisfies the Hölder conditions i.e. for any two values of variable \( x \) from the interval \([-1, 1]\), the following inequality is true,

\[ |f(x_1) - f(x_2)| \leq K|x_1 - x_2|^\alpha, \]

where \( 0 < \alpha \leq 1 \), \( x_1, x_2 \in [-1, 1] \), \( K \) is a Hölder constant.

2. \( C^1[-1, 1] \) is a class of function \( f(x) \), at which first derivative of the function is continuous on \([-1, 1]\).

**Theorem 1** Let \( f(t) \in H^\alpha([-1, 1], K) \), and \( E \) be a set of canonic partition of the interval \([-1, 1]\). Then for the error of quadrature formula (10) the following estimation of error is valid

\[ |R_N(t_j + \epsilon)| \leq \begin{cases} 
L_1 h^\alpha \ln(N + 1), & \text{when } \epsilon = h/2, \\
L_1 h^\alpha \ln(N + 1) + L_2 h^\delta, & \text{when } \epsilon \neq h/2, \\
L_1 h^\alpha \ln(N + 1), & \text{when } \epsilon = 0,
\end{cases} \]

where

\[ L_1 = 8K \left( 1 + \frac{1.506}{\alpha \ln(N + 1)} \right), \quad L_2 = K(0.068h^{5-\delta} + 0.567h^{2-\delta} + 0.516). \]

and

\[ 0 < \delta \leq \log_h \left( \frac{|2\epsilon - h|h}{(2h - \epsilon)(h + \epsilon)} \right). \]
**Theorem 2** Let \( f(t) \in C^1([-1, 1]) \), and \( E \) be a set of canonic partition of the interval \([-1, 1]\). Then the error of quadrature formula (10) can be written as

\[
|R_N(t_j + \varepsilon)| \leq \begin{cases} 
L_1^* h \ln(N + 1), & \text{when } \varepsilon = h/2, \\
L_1^* h \ln(N + 1) + L_2^* h, & \text{when } \varepsilon \neq h/2, \\
L_1^* h \ln(N + 1), & \text{when } \varepsilon = 0,
\end{cases}
\]

where

\[
L_1^* = 8M_1 \left(1 + \frac{0.78}{\ln(N + 1)}\right),
\]

\[
L_2^* = M_1 \left(0.068 h^{5-\delta} + 0.567 h^{2-\delta} + 0.516\right) \text{ and } M_1 = \max_{-1 \leq \theta \leq 1} |f'(\theta)|
\]

In the sequel we use the following well known results.

**Theorem 3 Euler-Makleron theorem.** Let \( f(x) \) be defined function on \([a, b]\). If \( f(x) \) is continuously differentiable function up to \(2k\), then the following formula is true

\[
\int_a^b f(x)dx = \frac{h}{2}(f(a) + f(b)) + \sum_{j=1}^{n-1} f(a + jh)h + R_{2k}(f),
\]

where

\[
R_{2k}(f) = -\frac{h^{2k}}{(2k)!} (b-a) B_{2k} f^{2k}(\xi), \quad \xi \in [a, b], h = \frac{b-a}{n},
\]

and \( B_{2k} \) is Bernoulli number. In addition if for any \( x \in [a, b] \) the following inequalities

\[
f^{(2k)}(x) \geq 0 \quad \text{and} \quad f^{(2k+2)}(x) \geq 0 \quad \text{or} \quad f^{(2k)}(x) \leq 0 \quad \text{and} \quad f^{(2k)}(x) \geq 0
\]

are true, then

\[
R_{2k}(f) = -\frac{h^{2k} B_{2k}}{(2k)!} [f^{2k-1}(b) - f^{2k-1}(a)].
\]

For the proof see [3].

The proof of theorems 1 and 2 are based on the following lemmas.

**Lemma 1** Let \( f(t) \) be continuous function on the interval \([-1, 1]\) and the function \( \varphi(t) \) be defined by (8).

If \( f(t) \in H^\alpha([-1, 1], K) \) then for any \( t', t'' \), \( t \in [-1, 1] \) the following estimations are true

a) \( |\varphi(t'') - \varphi(t')| \leq 2K|t'' - t'|^\alpha \),

b) \( |\varphi(t)| \leq K(1 - t^2)^\alpha \).
If \( f(t) \in C^1([-1, 1]) \), then for any \( t', t'' \), \( t \in [-1, 1] \), the following estimations are valid

c) \( |\varphi(t'') - \varphi(t')| \leq 2M_1|t'' - t'| \),

d) \( |\varphi(t)| \leq M_1(1 - t^2) \).

where \( M_1 = \max_{\xi \in [-1, 1]} |f'(\xi)| \).

Lemma 1 is proved in [4].

Since \( x \in [t_j, t_{j+1}] \), the following Lemma 2 is obvious.

**Lemma 2** Let \( x \) be any singular point in the interval \([t_j, t_{j+1}], j = 1, \ldots, N-1\). Then the following statement is true

\[
\int_{t_j}^{t_{j+1}} \frac{dt}{|t-x|} + \int_{t_{j+1}}^{t_{j+2}} \frac{dt}{|t-x|} \leq 2 \ln(N+1).
\]

**Lemma 3** Let \([x_1, x_2]\) be any interval on \( R \). If \( a \in (x_1, x_2) \) and \( 0 \leq \beta \leq 1 \), then the following statements are valid

\( a \) \( r_1^\beta + r_2^\beta \leq 2^{1-\beta}(r_1 + r_2)^\beta \),

\( b \) \( |r_1^\beta - r_2^\beta| \leq |r_1 - r_2|^\beta \),

where \( r_1 = |x_1 - a| \), \( r_2 = |x_2 - a| \).

The proof of Lemma 3 is given in [5].

Let \( g_{1, \nu}(x) \) be given by

\[
g_{1, \nu}(t) = f(t) - s_{1, \nu}(t), \quad \nu = j - 1, j, j + 1
\]

where \( t \in [t_\nu, t_{\nu+1}] \).

**Lemma 4** Let \( f(x) \) be continuous function and \( x \) be any point of the interval \([t_j, t_{j+1}]\). Then the following estimations are true

\( a \) If \( f(t) \in H^\alpha(K, [t_j, t_{j+1}]) \), then \( |g_{1, j}(x)| \leq 2\alpha K \left( |x - t_j||x - t_{j+1}| \right)^\alpha h^{-\alpha} \).

\( b \) If \( f(t) \in C^1([t_j, t_{j+1}]) \), then \( |g_{1, j}(x)| \leq M_1 \left( |x - t_j||x - t_{j+1}| \right) h^{-1} \),

where \( M_1 = \max |f'(c_1) - f'(c_2)| \), \( c_1 \in (t_j, x) \) and \( c_2 \in (x, t_{j+1}) \).

**Proof Lemma 4a** Since \( x \in [t_j, t_{j+1}] \), from (2) and (17) it follows that

\[
|g_{1, j}(x)| = \frac{1}{h} \left| (t_{j+1} - x)(f(x) - f(t_j)) + (x - t_j)(f(x) - f(t_{j+1})) \right|
\]

\[
\leq \frac{K}{h} \left( |x - t_j||x - t_{j+1}| \right)^\alpha (|x - t_{j+1}|^{1-\alpha} + |x - t_j|^{1-\alpha})
\]

\[
\leq 2\alpha K \left( |x - t_j||x - t_{j+1}| \right)^\alpha h^{-\alpha}.
\]

**Proof Lemma 4b** Due to (2), (17) and the Mean Value Theorem one has

\[
|g_{1, j}(x)| = \frac{1}{h} \left| (t_{j+1} - x)f'(c_1)(x - t_j) + (x - t_j)f'(c_2)(x - t_{j+1}) \right|
\]

\[
\leq M_1 \left( |x - t_j||x - t_{j+1}| \right) h^{-1}.
\]
Lemma 5 Let $f(t)$ be continuous function, and for any $x \in [t_j, t_{j+1}]$ the following inequalities are valid

a) If $f(t) \in H^\alpha(K, [t_{j-1}, t_{j+2}])$, then
\[
|s_{1,\nu}(t) - s_{1,j}(x)| \leq Kh^{\alpha-1}|t - x|, \quad \text{for all } \nu = j - 1, j, j + 1.
\]

b) If $f(t) \in C^1([t_{j-1}, t_{j+2}])$, then
\[
|s_{1,\nu}(t) - s_{1,j}(x)| \leq M_1|t - x|, \quad \text{for all } \nu = j - 1, j, j + 1,
\]
where $M_1 = \max_{t_{j-1} \leq \theta \leq t_{j+2}} |f'(\theta)|$.

Proof Lemma 5a) Let $\nu = j - 1$, using (2) and applying Lemma 3a) one has
\[
|s_{1,j-1}(t) - s_{1,j}(x)| = \frac{1}{h}|(t - t_j)(f(t_j) - f(t_{j-1})) + (x - t_j)(f(t_j) - f(t_{j+1}))|
\leq Kh^{\alpha-1}|t - t_j| + |x - t_j| \leq Kh^{\alpha-1}|t - x|.
\]

The case $\nu = j + 1$ is proved analogously. For $\nu = j$, points $x$ and $t$ belong to $[t_j, t_{j+1}]$. Then using (2) one gets
\[
|s_{1,j}(t) - s_{1,j}(x)| = \frac{1}{h}|(x - t)(f(t_j) - f(t_{j+1})| \leq Kh^{\alpha-1}|t - x|.
\]

Lemma 5b) is proved in the same manner.

Lemma 6 Let $f(t)$ be a continuous function on $[-1, 1]$, and $x$ be any point of the interval $[t_j, t_{j+1}]$.

a) If $f(t) \in H^\alpha(K, [t_{j-1}, t_{j+1}])$, then
\begin{enumerate}
  \item[(i)] $|g_{1,\nu}(t) - g_{1,j}(x)| \leq 3K|t - x|^{\alpha}$, for $\nu = j - 1, j, j + 1$,
  \item[(ii)] $|g_{1,\nu}(t) - g_{1,j}(x)| \leq 2K|t - x|^{\alpha}$ for $\nu = j$.
\end{enumerate}

b) If $f(t) \in C^1([-1, 1])$, then
\[
|g_{1,\nu}(t) - g_{1,j}(x)| \leq 2M_1|t - x|, \quad \text{for all } \nu = j - 1, j, j + 1
\]
where $M_1 = \max_{\xi \in [t_{j-1}, t_{j+2}]} |f'(\xi)|$.

Proof Lemma 6a(i) Since $x \in [t_j, t_{j+1}]$, $t \in [t_{j-1}, t_j]$ and $\nu = j - 1$, due to Lemma 5a) we obtain
\[
|g_{1,j-1}(t) - g_{1,j}(x)| = |f(t) - f(x) - (s_{1,j-1}(t) - s_{1,j}(x))|
\leq K|t - x|^{\alpha} + Kh^{\alpha-1}|t - x| \leq 3K|t - x|^{\alpha}.
\]

The case $\nu = j + 1$ is asserted in a similar way.
Proof Lemma 6a(ii) In the case \( t, x \in [t_j, t_{j+1}] \) and \( \nu = j \), in view of Lemma 5a) one has

\[
|g_{1,j}(t) - g_{1,j}(x)| = |f(t) - f(x) - (s_{1,j}(t) - s_{1,j}(x))| \\
\leq K|t - x|^{\alpha} + Kh^{\alpha - 1}|t - x| \leq 2K|t - x|^{\alpha}.
\]

Proof Lemma 6b) For the case \( \nu = j - 1 \) the point \( t \in [t_{j-1}, t_j] \). Then due to Lemma 5a) for any \( x \in [t_j, t_{j+1}] \) we obtain

\[
|g_{1,j-1}(t) - g_{1,j}(x)| = |f(t) - f(x) - (s_{1,j-1}(t) - s_{1,j}(x))| \\
\leq |f'(c_1)||t - x| + M_1|t - x| \leq 2M_1|t - x|.
\]

Let \( R^*(t_j + \varepsilon) \) be defined by

\[
R^*(t_j + \varepsilon) = (1 - (t_j + \varepsilon)^2) \alpha \left[ \int_{t_{j-1}}^{t_j} \frac{dt}{t - (t_j + \varepsilon)} + \int_{t_{j+2}}^{t_j} \frac{dt}{t - (t_j + \varepsilon)} \right] \\
- \sum_{k=1,k \notin Q(j),k \neq j+2}^N \frac{h}{t_k - (t_j + \varepsilon)}.
\]

(18)

For \( \varepsilon = 0 \) we consider the following expressions

\[
R^*(t_j) = (1 - t_j^2)^\alpha \left[ \int_{t_{j-1}}^{t_j} \frac{dt}{t - t_j} + \int_{t_{j+2}}^{t_j} \frac{dt}{t - t_j} - \sum_{k=1,k \notin Q(j),k \neq j}^N \frac{h}{t_k - t_j} \right].
\]

Lemma 7 For any singular point \( (t_j + \varepsilon), \ j = 1, ..., N \) where \( t_j \in E \) the following statement is valid

\[
|R^*(t_j + \varepsilon)| \leq \begin{cases} C_1h^\alpha, & \text{when } \varepsilon = h/2, \\ C_2h^\delta, & \text{when } \varepsilon \neq h/2, \\ C_1h^\alpha, & \text{when } \varepsilon = 0, \end{cases}
\]

where

\[
C_1 = 0.5 + 0.067h, \ \text{and} \ \ C_2 = 0.068h^{5-\delta} + 0.567h^{2-\delta} + 0.516.
\]

Proof Lemma 7 Let us examine the following function

\[
f(t) = \frac{1}{t - (t_j + \varepsilon)}.
\]

(19)
where $t$ runs either $[-1, t_{j-1}]$ or $[t_{j+2}, 1]$. It is obvious that second and fourth derivatives of the function (19) are negative at $t \in [-1, t_{j-1}]$ and both positive on $[t_{j+2}, 1]$. Applying Euler-Makleron formula (16) we obtain

$$
\int_{-1}^{t_{j-1}} \frac{dt}{t - (t_j + \varepsilon)} = \sum_{k=1}^{j-2} \frac{h}{t_k - (t_j + \varepsilon)} - \frac{h}{2} \left( \frac{1}{1 + (t_j + \varepsilon)} + \frac{1}{h + \varepsilon} \right) + h^4 \left[ \frac{-1}{(h + \varepsilon)^4} + \frac{1}{1 + (t_j + \varepsilon))^4} \right],
$$

and

$$
\int_{t_{j+2}}^{1} \frac{dt}{t - (t_j + \varepsilon)} = \sum_{k=j+3}^{N} \frac{h}{t_k - (t_j + \varepsilon)} + \frac{h}{2} \left( \frac{1}{2h - \varepsilon} - \frac{1}{1 - (t_j + \varepsilon)} \right) + h^4 \left[ \frac{-1}{(1 - (t_j + \varepsilon))^4} + \frac{1}{2h - \varepsilon} \right].
$$

Substituting these formulas into (18) and letting $A(\varepsilon) = \frac{|2\varepsilon - h|h}{2(2h - \varepsilon)(h + \varepsilon)}$ yields

$$
R^*(t_j + \varepsilon) \leq (1 - (t_j + \varepsilon)^2)^{\alpha} \left[ \frac{|t_j + \varepsilon|h}{2(1 - (t_j + \varepsilon)^2)} + \frac{|t_j + \varepsilon|h^4}{15(1 - (t_j + \varepsilon)^2)^3} \right] + A(\varepsilon) \left[ 1 + \frac{h^4(5h^2 - 2h\varepsilon + 2\varepsilon^2)}{20(2h - \varepsilon)(h + \varepsilon)^3} \right]. \tag{20}
$$

If $\varepsilon = h/2$, then $A(\varepsilon) \equiv 0$ hence

$$
R^*(t_{0j}) \leq (1 - t_{0j}^2)^{\alpha} \left[ \frac{|t_{0j}|h}{2(1 - t_{0j}^2)} + \frac{|t_{0j}|h^4}{15(1 - t_{0j}^2)^3} \right].
$$

The expressions on the right hand side attains its maximum value at $j = N$. We therefore have

$$
R^*(t_{0N}) \leq (1 - t_{0N}^2)^{\alpha} \left[ \frac{|t_{0N}|h}{2(1 - t_{0N}^2)} + \frac{|t_{0N}|h^4}{15(1 - t_{0N}^2)^3} \right] \leq \frac{h^{\alpha}}{2} + \frac{1}{15} h^{1+\alpha} = C_1 h^{\alpha}.
$$

Let $\varepsilon \neq h/2$, then on the right hand side of (20) is reached its max value at $t_j = 0$. Thus

$$
R^*(\varepsilon) \leq (1 - \varepsilon^2)^{\alpha} \left[ \frac{\varepsilon h}{2(1 - \varepsilon^2)} + \frac{\varepsilon h^4}{15(1 - \varepsilon^2)^3} \right] + \frac{|2\varepsilon - h|h}{2(2h - \varepsilon)(h + \varepsilon)} \left[ 1 + \frac{h^4(5h^2 - 2h\varepsilon + 2\varepsilon^2)}{20(2h - \varepsilon)(h + \varepsilon)^3} \right] \leq 0.567 h^2 + 0.068 h^5 + 0.516 \frac{|2\varepsilon - h|h}{(2h - \varepsilon)(h + \varepsilon)}.
$$
Since $0 < \varepsilon < h$ there is a $\delta$ such that

$$\frac{|2\varepsilon - h|h}{(2h - \varepsilon)(h + \varepsilon)} < h^\delta,$$

it follows that

$$0 < \delta \leq \log h \frac{|2\varepsilon - h|h}{(2h - \varepsilon)(h + \varepsilon)}.$$

Thus we have the following error

$$R^*(\varepsilon) \leq C_2 h^\delta.$$

**Proof Theorem 1** From (7) it follows that

$$|R_N(t_j + \varepsilon)| = \left| \int_{t_j}^{t_j + h} \frac{\varphi(t)dt}{t - (t_j + \varepsilon)} - \sum_{k=1}^{N} \frac{\varphi(t_k)h}{t_k - (t_j + \varepsilon)} - \sum_{\nu=j-1}^{j-1} \int_{t_{\nu}}^{t_{\nu+1}} \frac{s_{1,\nu}(t)dt}{t - (t_j + \varepsilon)} \right| \quad (21)$$

where

$$R_1(t_j + \varepsilon) = \left| \int_{t_j}^{t_j + h} \frac{\varphi(t) - \varphi(t_j + \varepsilon)}{t - (t_j + \varepsilon)} dt \right|,$$

$$R_2(t_j + \varepsilon) = \left| \frac{\varphi(t_j + \varepsilon) - \varphi(t_j + \varepsilon)}{t_j + \varepsilon - (t_j + \varepsilon)} \right|,$$

$$R_3(t_j + \varepsilon) = \left| \sum_{k=1, k \not\in \Omega(\varphi)} \left( \int_{t_j}^{t_j + h} \frac{\varphi(t) - \varphi(t_j + \varepsilon)}{t - (t_j + \varepsilon)} - \frac{\varphi(t_k) - \varphi(t_j + \varepsilon)}{t_k - (t_j + \varepsilon)} \right) dt \right|,$$

$$R_4(t_j + \varepsilon) = \left| \varphi(t_j + \varepsilon) \left( \int_{t_j}^{t_j + h} \frac{dt}{t - (t_j + \varepsilon)} - \sum_{k=1, k \not\in \Omega(\varphi)} \frac{h}{t_k - (t_j + \varepsilon)} \right) \right|,$$

$$R_5(t_j + \varepsilon) = \left| \sum_{\nu=j-1}^{j-1} \int_{t_{\nu}}^{t_{\nu+1}} \frac{f(t) - s_{1,\nu}(t)}{t - (t_j + \varepsilon)} dt \right|.$$

Due to Lemma 1a) one gets

$$R_1(t_j + \varepsilon) \leq 2K \int_{t_j}^{t_j + h} \frac{dt}{|t - (t_j + \varepsilon)|^{1-\alpha}} \leq \frac{2K}{\alpha} h^\alpha,$$

$$R_2(t_j + \varepsilon) \leq 2K \frac{h}{|t_{j+2} - (t_j + \varepsilon)|^{1-\alpha}} \leq 2Kh^\alpha.$$

For $R_3(t_j + \varepsilon)$ we write

$$\frac{\varphi(t) - \varphi(t_j + \varepsilon)}{t - (t_j + \varepsilon)} - \frac{\varphi(t_k) - \varphi(t_j + \varepsilon)}{t_k - (t_j + \varepsilon)} = \frac{\varphi(t) - \varphi(t_k)}{t - (t_j + \varepsilon)} + \frac{\varphi(t_k) - \varphi(t_j + \varepsilon)}{t_k - (t_j + \varepsilon)} \frac{t_k - t}{t - (t_j + \varepsilon)}.$$

$$\frac{\varphi(t) - \varphi(t_k)}{t - (t_j + \varepsilon)} + \frac{\varphi(t_k) - \varphi(t_j + \varepsilon)}{t_k - (t_j + \varepsilon)} \frac{t_k - t}{t - (t_j + \varepsilon)},$$
and applying Lemma 1a) and Lemma 2 yields

\[ R_3(t_j + \varepsilon) \leq 8K h^\alpha \ln(N + 1). \]

Due to Lemma 1b) and Lemma 7, for \( R_4 \) we obtain

\[
|R_4(t_j + \varepsilon)| \leq \begin{cases} 
K(0.5 + 0.067h)h^\alpha, & \text{when } \varepsilon = \frac{h}{2}, \\
K(0.567h^{2-\delta} + 0.068h^{5-\delta} + 0.516)h^\delta, & \text{when } \varepsilon \neq h/2, \\
K(0.5 + 0.067h)h^\alpha, & \text{when } \varepsilon = 0.
\end{cases}
\]

To obtain \( R_5 \) we use the formula (17) and Lemma 3, Lemma 4a) and Lemma 6a) hence

\[
R_5(t_j + \varepsilon) \leq \left| \sum_{\nu=j-1}^{j+1} \frac{g_1,\nu(t) - g_1,\nu(t_j + \varepsilon)}{t - (t_j + \varepsilon)} dt \right| + \left| \sum_{\nu=j-1}^{j+1} g_1,\nu(t_j + \varepsilon) \int_{t_j}^{t_{\nu+1}} \frac{dt}{t - (t_j + \varepsilon)} \right| \\
\leq 3K \int_{t_{j-1}}^{t_j} \frac{dt}{|t - (t_j + \varepsilon)|^{1-\alpha}} + 2K \int_{t_j}^{t_{j+1}} \frac{dt}{|t - (t_j + \varepsilon)|^{1-\alpha}} \\
+ 3K \int_{t_{j+1}}^{t_{j+2}} \frac{dt}{|t - (t_j + \varepsilon)|^{1-\alpha}} + 2^\alpha K h^{-\alpha}(\varepsilon(h-\varepsilon))^\alpha \left| \ln \frac{t_{j+2} - (t_j + \varepsilon)}{t_{j-1} - (t_j + \varepsilon)} \right| \\
\leq \frac{10K}{\alpha} h^\alpha + 0.3365 \cdot 2^{-3\alpha} K h^\alpha \leq \frac{10.012K}{\alpha} h^\alpha.
\]

Substituting \( R_1-R_5 \) into (21) proves Theorem 1. The case \( \varepsilon = 0 \) is proved with the use of QF (14), Euler-Makleron formula (16) and corresponding Lemma 1-7. Theorem 2 is proved in the same manner.

4 Numerical experiments

Let \( f(t) = t^2 - 3t + 5 \). Then exact solution for the singular integral (1) is given by

\[ J(x) = (x^2 - 3x + 5)\ln\frac{1-x}{1+x} + 2x - 6 \tag{22} \]

Discrete vortex method (MDV) is computed as follows

\[ \tilde{J}(t_j + \varepsilon) = \sum_{k=1}^{N} \frac{f(t_k)dt}{t_k - (t_j + \varepsilon)} \tag{23} \]

In the following tables we give some numerical results for QF which is computed by (10) in comparison with discrete vortex method (23).
Table 1.
\[ N = 19, \ h = 0.1, \ \varepsilon = h/2 = 0.05 \]

<table>
<thead>
<tr>
<th>( j )</th>
<th>( x )</th>
<th>( \text{Exact} )</th>
<th>( (10) )</th>
<th>( (21) )</th>
<th>Error ( (10) )</th>
<th>Error ( (23) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.85</td>
<td>13.083048</td>
<td>13.243225</td>
<td>16.294800</td>
<td>0.160177</td>
<td>3.211752</td>
</tr>
<tr>
<td>2</td>
<td>-0.75</td>
<td>7.702423</td>
<td>7.846796</td>
<td>9.517194</td>
<td>0.144373</td>
<td>1.814771</td>
</tr>
<tr>
<td>5</td>
<td>-0.45</td>
<td>-0.548003</td>
<td>-0.459877</td>
<td>0.182127</td>
<td>0.088126</td>
<td>0.730130</td>
</tr>
<tr>
<td>9</td>
<td>-0.05</td>
<td>-5.584320</td>
<td>-5.574487</td>
<td>-5.252632</td>
<td>0.009833</td>
<td>0.331688</td>
</tr>
<tr>
<td>10</td>
<td>0.05</td>
<td>-6.385655</td>
<td>-6.395488</td>
<td>-6.385655</td>
<td>0.009833</td>
<td>0.2698655</td>
</tr>
<tr>
<td>11</td>
<td>0.15</td>
<td>-7.082179</td>
<td>-7.111669</td>
<td>-6.869733</td>
<td>0.029490</td>
<td>0.212446</td>
</tr>
<tr>
<td>15</td>
<td>0.55</td>
<td>-9.417276</td>
<td>-9.524661</td>
<td>-9.473978</td>
<td>0.107385</td>
<td>0.056703</td>
</tr>
<tr>
<td>17</td>
<td>0.75</td>
<td>-10.9458274</td>
<td>-11.090200</td>
<td>-11.331290</td>
<td>0.144373</td>
<td>0.385463</td>
</tr>
<tr>
<td>18</td>
<td>0.85</td>
<td>-12.270290</td>
<td>-12.430467</td>
<td>-13.139348</td>
<td>0.160177</td>
<td>0.869058</td>
</tr>
</tbody>
</table>

Table 2.
\[ N = 99, \ h = 0.01, \ \varepsilon = 3h/10 = 0.006, \ x = t_j + \varepsilon \]

<table>
<thead>
<tr>
<th>( j )</th>
<th>( x )</th>
<th>( \text{Exact} )</th>
<th>( (10) )</th>
<th>( (23) )</th>
<th>Error ( (10) )</th>
<th>Error ( (23) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.974</td>
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<td>30.4924675</td>
<td>43.742878</td>
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<tr>
<td>2</td>
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<td>24.978597</td>
<td>25.008975</td>
<td>36.371769</td>
<td>0.030379</td>
<td>11.393172</td>
</tr>
<tr>
<td>10</td>
<td>-0.794</td>
<td>9.753531</td>
<td>9.758610</td>
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</tr>
<tr>
<td>30</td>
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<td>-1.508749</td>
<td>-1.553432</td>
<td>5.353967</td>
<td>0.044683</td>
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</tr>
<tr>
<td>49</td>
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<td>-5.886809</td>
<td>-5.957720</td>
<td>0.465924</td>
<td>0.070911</td>
<td>5.420886</td>
</tr>
<tr>
<td>50</td>
<td>0.006</td>
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<td>-6.119505</td>
<td>-0.693341</td>
<td>0.071720</td>
<td>5.354444</td>
</tr>
<tr>
<td>51</td>
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<td>-6.276508</td>
<td>-0.915147</td>
<td>0.072471</td>
<td>5.288890</td>
</tr>
<tr>
<td>60</td>
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<td>-7.513989</td>
<td>-2.699397</td>
<td>0.076599</td>
<td>4.737933</td>
</tr>
<tr>
<td>80</td>
<td>0.606</td>
<td>-9.775212</td>
<td>-9.844543</td>
<td>-6.030498</td>
<td>0.069330</td>
<td>3.744715</td>
</tr>
<tr>
<td>90</td>
<td>0.806</td>
<td>-11.597817</td>
<td>-11.654772</td>
<td>-8.281660</td>
<td>0.056954</td>
<td>3.316157</td>
</tr>
<tr>
<td>97</td>
<td>0.946</td>
<td>-15.065660</td>
<td>-15.109793</td>
<td>-12.374845</td>
<td>0.044133</td>
<td>2.690815</td>
</tr>
<tr>
<td>98</td>
<td>0.966</td>
<td>-16.382829</td>
<td>-16.424344</td>
<td>-14.093328</td>
<td>0.041515</td>
<td>2.289501</td>
</tr>
</tbody>
</table>

4.1 Conclusion

In Table 1 where the singular point \( x \) is located in the middle of the subinterval \([t_j, t_{j+1}]\), both methods show good convergence. Nevertheless quadrature formula (10) is better than (23) in terms of erroneous. Whereas Table 2 shows that when singular point \( x \) is not located in the middle of the interval, MDV (21) does not converge but QF(10) still provides convergence. Hence our QF provides convergence for any singular point in the interval \((-1, 1)\).
References


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