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# Equivalence and Evaluation of Certain Laplace Transforms

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**Abstract**

In this paper, certain Laplace transforms of the product of two $\Psi$ functions and their equivalent forms in terms of Gauss functions are obtained. A further specialization of these results yields explicit representations for the product of two $\Psi$ , Bessel, parabolic cylinder and confluent hypergeometric functions.

**Mathematics Subject Classifications:** 33C05; 33C15; 33C70

**Keywords:** $\Psi$ function; Gauss hypergeometric function; confluent hypergeometric function; Laplace transform

## 1. Introduction

The function

$$\Psi[b,c;x]=\frac{1}{\Gamma(b)}\int_0^\infty \exp(-xt)\,t^{b-1}\,(1+t)^{c-b-1}dt \qquad (\,\mathrm{Re}(b),\mathrm{Re}(x)>0\,), \quad (1.1)$$

defines an alternative form of solution of Kummer's equation ([8]; p.36(1)) in the half plane : $\mathrm{Re}(x)>0$.

The following relation between the confluent hypergeometric function ${}_1F_1$ ([8]; p.36(3)) and $\Psi$ functions is worthy of note :

$$\Psi[b,c;x]=\frac{\Gamma(1-c)}{\Gamma(b-c+1)}\,{}_1F_1[b;\,c;\,x]+\frac{\Gamma(c-1)}{\Gamma(b)}\,x^{1-c}\,{}_1F_1[b-c+1;\,2-c;\,x]$$

$$(\;\; |x|<\infty\;; \quad c\neq 0,\pm 1,\pm 2,...\;\;). \qquad (1.2)$$

Further, we note the following expressions of Whittaker functions of the first and second kind $M_{k,\,\mu}(x)$ and $W_{k,\,\mu}(x)$ respectively in terms of ${}_1F_1$ and $\Psi$ functions :

$$\begin{aligned} M_{k,\,\mu}(x) &= x^{\mu+\frac{1}{2}}\exp(-\frac{1}{2}\,x)\,{}_1F_1\left[\mu-k+\frac{1}{2};\,2\mu+1;\,x\right] \\ &= x^{\mu+\frac{1}{2}}\exp(\frac{1}{2}\,x)\,{}_1F_1\left[\mu+k+\frac{1}{2};\,2\mu+1;\,-x\right] \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} W_{k,\,\mu}(x) &= x^{\mu+\frac{1}{2}}\exp(-\frac{1}{2}\,x)\,\Psi\left[\mu-k+\frac{1}{2};\,2\mu+1;\,x\right] \\ &= W_{k,\,-\mu}(x). \end{aligned} \tag{1.4}$$

We shall study here the following Laplace transform of the product of two $\Psi$ functions

$$I_1 = \int_0^\infty t^{\lambda+a+b-c-1}\exp(-yt)\,\Psi[a,\,a+b-c+1;\,wt]\,\Psi[b,\,a+b-c+1;\,zt]\,dt\,. \tag{1.5}$$

It is shown that this integral can be expressed in the form

$$I_1 = \frac{\Gamma(\lambda)}{\Gamma(c)}\int_0^\infty \frac{t^{c-1}}{(t+w)^a(t+z)^b(t+y)^\lambda}\,{}_2F_1\left[a,\,b;\,c;\,\frac{t(t+w+z)}{(t+z)(t+w)}\right]dt\,, \tag{1.6}$$

where ${}_2F_1$ is Gauss hypergeometric function ([8]; p.29(4)). The integral (1.5) is of interest for the physicists in connection with transition matrix elements with either in-or outgoing fields. In a series of papers Olsson [3-6] discussed particular cases of this integral and derived results expressed in terms of various functions connected with Appell's hypergeometric function $F_2$ of two variables ([8]; p.53(5)):

$$F_2[a,\,b_1,\,b_2;\,c_1,\,c_2;\,x,y] = \sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,(b_1)_m\,(b_2)_n}{(c_1)_m\,(c_2)_n\,m!\,n!}\,x^m y^n \quad (\ |x|+|y|<1\ ). \tag{1.7}$$

In order to extend the integral (1.5), we investigate the Laplace transform of the product of two $\Psi$ functions with Whittaker fuction $M_{k,\mu}(x)$

$$I_2 = \int_0^\infty t^{\nu+a+b-c-1}\exp(-yt)\,M_{k,\,\mu}(\delta t)\,\Psi[a,\,a+b-c+1;\,wt]\,\Psi[b,\,a+b-c+1;\,zt]dt. \tag{1.8}$$

In particular the above integral can be expressed as an integral involving product of Gauss hypergeometric function ${}_2F_1$ in the form

$$I_2 = \frac{\delta^{\mu+\frac{1}{2}}\Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(c)} \int_0^\infty \frac{t^{c-1}}{(t+w)^a(t+z)^b(t+y+\frac{1}{2}\delta)^{\mu+\nu+\frac{1}{2}}}$$

$$\times\, {}_2F_1\left[a,\, b;\, c;\, \frac{t(t+w+z)}{(t+z)(t+w)}\right] {}_2F_1\left[\mu+\nu+\frac{1}{2},\, \mu-k+\frac{1}{2};\, 2\mu+1;\, \frac{\delta}{t+y+\frac{1}{2}\delta}\right] dt. \tag{1.9}$$

By specializing the parameters in integral (1.9), one may derive integrals involving Legendre function, Jacobi polynomials and many other functions.

In section 2, we establish the following Laplace transform of Gauss hypergeometric function

$$I' = \int_0^\infty \frac{t^{\lambda-1}\exp(-pt)}{(t+w)^a(t+z)^b}\, {}_2F_1\left[\alpha,\, \beta;\, \gamma;\, \frac{t(t+w+z)}{(t+w)(t+z)}\right] dt$$

$$= \Gamma(\lambda-a-b)\, p^{a+b-\lambda} \sum_{n=0}^\infty \frac{(\alpha)_n\, (\beta)_n}{(\gamma)_n\, n!}$$

$$\times\, \Phi_2^{(3)}[a+n,\, b+n,\, -n;\, a+b-\lambda+1;\, pw, pz,\, p(w+z)] \quad (\,\mathrm{Re}(p), \mathrm{Re}(\lambda-a-b) > 0\,)\,, \tag{1.10}$$

where $\Phi_2^{(n)}$ is confluent hypergeometric function of $n$ variables ([8]; p.62(10)) defined by

$$\Phi_2^{(n)}[b_1, b_2, \ldots, b_n;\, c;\, x_1, x_2, \ldots, x_n]$$
$$= \sum_{m_1,m_2,\ldots,m_n=0}^\infty \frac{(b_1)_{m_1}\,(b_2)_{m_2}\ldots(b_n)_{m_n}}{(c)_{m_1+m_2+\ldots+m_n}} \frac{x_1^{m_1}\, x_2^{m_2}\ldots x_n^{m_n}}{m_1!\, m_2!\, \ldots m_n!}\,. \tag{1.11}$$

In Section 3, we derive equivalence of integrals (1.5) and (1.6), and of integrals (1.8) and (1.9). In Section 4, it is shown how further specialization of integral (1.10) enables us to obtain the representations of the product of two $\Psi$, Bessel, parabolic cylinder and confluent hypergeometric functions. Finally, we give some concluding remarks in section 5 .

## 2. Proof of Laplace Transform (1.10)

We use the definition of Gauss hypergeometric function ${}_2F_1$ ([8]; p.29(4)) in the l.h.s of Eq. (1.10), to get

$$I' = \sum_{n=0}^\infty \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n\, n!} \int_0^\infty \exp(-pt)\, t^{\lambda+n-1}\,(t+w)^{-a-n}\,(t+z)^{-b-n}\,(t+w+z)^n\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n\,n!} \int_0^{\infty} \exp(-pt)\,t^{\lambda-a-b-1}\left(1+\frac{w}{t}\right)^{-a-n}\left(1+\frac{z}{t}\right)^{-b-n}\left(1+\frac{w+z}{t}\right)^{n} dt\,, \tag{2.1}$$

which on using binomial expansion and inverting the order of summation and integration, gives

$$I' = \sum_{n,s,r=0}^{\infty}\sum_{k=0}^{n} \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n\,n!}(-1)^{s+r+k}\,(a+n)_s\,(b+n)_r\,(-n)_k\,\frac{w^s\,z^r\,(w+z)^k}{s!\;r!\;k!}$$

$$\times \int_0^{\infty} t^{\lambda-a-b-s-r-k-1}\exp(-pt)\;dt\;. \tag{2.2}$$

Further, by making use of Euler's integral ([1]; p.137(1)) in Eq.(2.2), simplifying and interpreting the resulting triple series in terms of confluent hypergeometric function of three variables $\Phi_2^{(3)}$ given by Eq.(1.11) ( for $n=3$) , we get the required result.

## 3. Equivalence of Integrals

First, we prove that integral (1.5) can be expressed in its equivalent form (1.6). We recall the results ([7]; p.321(17))

$$L[g(t);p] \;=\; L\left[\frac{t^{c-1}}{(t+w)^a\,(t+z)^b}\,{}_2F_1\left[a,b;\,c;\,\frac{t(t+w+z)}{(t+z)\,(t+w)}\right];\,p\right]$$

$$= \frac{\Gamma(c)}{p^{c-a-b}}\,\Psi[a,\,a+b-c+1;\,pw]\,\Psi[b,\,a+b-c+1;\,pz] \;=\; \xi(p)$$

$$(\ \mathrm{Re}(c),\mathrm{Re}(p)>0\ \ ;\ \ |\arg w|,|\arg z|<\pi\ ) \tag{3.1}$$

and ([1]; p.129(5), p.137(1))

$$L[h(t);p] \;=\; L[t^{\lambda-1}\exp(-yt);p] \;=\; \eta(p) \qquad (\ Re(p+y)\ ,\ \mathrm{Re}(\lambda)>0\ ), \tag{3.2}$$

where the Laplace transform of $f(t)$ is denoted as

$$\int_0^{\infty}\exp(-pt)f(t)\,dt \;=\; L[f(t);p]\;.$$

Now using pairs (3.1) and (3.2) in Goldstein theorem [2]

$$\int_0^{\infty} g(t)\,\eta(t)\,dt = \int_0^{\infty} h(t)\xi(t)\,dt\;, \tag{3.3}$$

we get the equivalence of integrals (1.5) and (1.6).

Next, we prove the equivalence of integrals (1.8) and (1.9). By using the well known property of the Laplace transform ([1]; p.129(5)) to the result ([1]; p.215(11)), we get

$$L[h(t);p] \;=\; L[t^{\nu-1}\exp(-yt)M_{k,\mu}(\delta t);p] \;=\; \frac{\delta^{\mu+\frac{1}{2}}\,\Gamma(\mu+\nu+\frac{1}{2})}{(y+p+\frac{1}{2}\delta)^{\mu+\nu+\frac{1}{2}}}$$

$$\times\,{}_2F_1\left[\mu+\nu+\frac{1}{2},\ \mu-k+\frac{1}{2};\ 2\mu+1;\ \frac{\delta}{(y+p+\frac{1}{2}\delta)}\right] \;=\; \sigma(p)$$

$$\left(\ Re(p+y)>\frac{1}{2}|\mathrm{Re}(\delta)|\ \ ;\ \ \mathrm{Re}(\mu+\nu)>-\frac{1}{2}\ \right)\ . \tag{3.4}$$

Now, using pairs (3.1) and (3.4) in Goldstein theorem (3.3), we get the equivalence of integrals (1.8) and (1.9).

## 4. Special cases

We consider the following special cases of integral (1.10):

I. Taking $\lambda=\gamma=c$ , $\alpha=a$ , $\beta=b$ in integral (1.10) and comparing the resultant integral with ([7]; p.321(17)), we get

$$\frac{\Gamma(c-a-b)}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}\,\Phi_2^{(3)}[a{+}n,\ b{+}n,\ -n;\ a{+}b{-}c{+}1;\ w,\ z,(w{+}z)]$$
$$=\ \Psi[a,\ a{+}b{-}c{+}1;\ w]\,\Psi[b,\ a{+}b{-}c{+}1;\ z]\ , \tag{4.1}$$

which , on taking $z=0$ gives

$$\frac{\Gamma(c-a)}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}\,\Phi_2^{(2)}[a{+}n,\ -n;\ a{+}b{-}c{+}1;\ w,\ w]=\Psi[a,\ a{+}b{-}c{+}1;\ w]\ . \tag{4.2}$$

Again taking $b=c-1$ in Eq.(4.1) and using relation (1.2) , we get

$$\frac{\Gamma(1-a)}{\Gamma(c-1)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!\,(c+n-1)}\,\Phi_2^{(3)}[a{+}n,\ c{+}n{-}1,\ -n;\ a;\ w,\ z,(w{+}z)]$$
$$=\ \left(\Gamma(1-a)\ \ \exp(w)+\frac{\Gamma(a-1)}{\Gamma(a)}\,w^{1-a}\,{}_1F_1[1;\ 2-a;\ w]\right)$$
$$\times\left(\frac{\Gamma(1-a)}{\Gamma(c-a)}\,{}_1F_1[c-1;\ a;\ z]\right.$$
$$\left.+\ \frac{\Gamma(a-1)}{\Gamma(c-1)}\,z^{1-a}\,{}_1F_1[c-a;\ 2-a;\ z]\right)\qquad(\ a\neq 0,\pm1,\pm2,...\ ). \tag{4.3}$$

II. Taking $\lambda = \gamma = 1,\ \alpha = \beta = b = a$ in integral (1.10) and using ([7]; p.321(18)) , we get

$$\sum_{n=0}^{\infty} \left(\frac{(a)_n}{n!}\right)^2 \Phi_2^{(3)}[a+n,\ a+n,\ -n;\ 2a;\ w, z, w+z]$$

$$= \frac{(wz)^{\frac{1}{2}-a}}{\pi\,\Gamma(1-2a)} \exp\bigl(\frac{1}{2}(w+z)\bigr)\, K_{a-\frac{1}{2}}\left(\frac{w}{2}\right)\ K_{a-\frac{1}{2}}\left(\frac{z}{2}\right), \tag{4.4}$$

where $K_\nu(z)$ is MacDonald function or modified Bessel function of the third kind ([7]; p.794] .

III. Taking $\lambda = \gamma = c\ ,\ \alpha = a\ ,\ \beta = b = c - a - \dfrac{1}{2}$ , in integral (1.10) and using ([7]; p.321(19)), we get

$$\sum_{n=0}^{\infty} \frac{(a)_n\ \left(c-a-\frac{1}{2}\right)_n}{(c)_n\, n!}\, \Phi_2^{(3)}\left[a+n,\ c-a+n-\frac{1}{2},\ -n;\ \frac{1}{2};\ w,\ z,\ w+z\right]$$

$$= \frac{2^{c-\frac{1}{2}}\,\Gamma(c)}{\sqrt{\pi}} \exp\bigl(\frac{1}{2}(w+z)\bigr)\, D_{-2a}(\sqrt{2w})\, D_{2a-2c+1}(\sqrt{2z})\,, \tag{4.5}$$

where $D_\nu(z)$ is the parabolic cylinder function ([7]; p.792).

## 5. Concluding Remarks

In this paper , we have shown the equivalence of integrals involving the product of two $\Psi$ functions with integrals involving hypergeometric function ${}_2F_1$ . Also, we have derived a Laplace transform for hypergeometric function in terms of confluent hypergeometric function of three variables , which on specializing the parameters yields representations of the product of two $\Psi$ functions , confluent hypergeometric functions ${}_1F_1$ , MacDonald functions $K_\nu(z)$ and the parabolic cylinder functions $D_\nu(z)$ . Further , these integrals can be expressed in terms of various functions connected with Appell's hypergeometric function $F_2$ , defined by Eq.(1.7) .

To give an example , we consider integral (1.8) . We express the Whittaker function $M_{k,\mu}(x)$ in terms of confluent hypergeometric function ${}_1F_1$ , using Eq.(1.3) in the integrand of (1.8) . Then integrating term-by-term using the result of Olsson ([4]; p.114(3)) and interpreting the resulting triple series in terms of Lauricella functions of three variables ([8]; p.60(1))

$$F_A^{(n)}[a, b_1, b_2, \ldots, b_n;\ c_1, c_2, \ldots, c_n;\ x_1, x_2, \ldots, x_n]$$

$$= \sum_{m_1,m_2,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+m_2+\dots+m_n}\,(b_1)_{m_1}(b_2)_{m_2}\dots(b_n)_{m_n}}{(c_1)_{m_1}(c_2)_{m_2}\dots(c_n)_{m_n}} \frac{x_1^{m_1}\,x_2^{m_2}\dots x_n^{m_n}}{m_1!\,m_2!\dots m_n!}$$

$$(\ |x_1|+|x_2|+\dots+|x_n|<1\ ), \tag{5.1}$$

we get

$$I_2 = \frac{\Gamma(\lambda+\mu+a+b-c+\frac{1}{2})(\Gamma(c-a-b))^2\,\delta^{\mu+\frac{1}{2}}}{\Gamma(c-a)\Gamma(c-b)(y+\frac{1}{2}\delta)^{\lambda+\mu+a+b-c+\frac{1}{2}}} F_A^{(3)}\left[\lambda+\mu+a+b-c+\frac{1}{2},\right.$$

$$\left.a,b,\mu-k+\frac{1}{2};a+b-c+1,a+b-c+1,2\mu+1;\frac{w}{y+\frac{1}{2}\delta},\frac{z}{y+\frac{1}{2}\delta},\delta\right]$$

$$+\frac{\Gamma(\lambda+\mu+\frac{1}{2})\Gamma(a+b-c)\Gamma(c-a-b)\delta^{\mu+\frac{1}{2}}z^{c-a-b}}{\Gamma(c-b)\Gamma(b)(y+\frac{1}{2}\delta)^{\lambda+\mu+\frac{1}{2}}}$$

$$\times F_A^{(3)}\left[\lambda+\mu+\frac{1}{2},a,c-a,\mu-k+\frac{1}{2};a+b-c+1,c-a-b+1,2\mu+1;\frac{w}{y+\frac{1}{2}\delta},\frac{z}{y+\frac{1}{2}\delta},\delta\right]$$

$$+\frac{\Gamma(\lambda+\mu+\frac{1}{2})\Gamma(a+b-c)\Gamma(c-a-b)\delta^{\mu+\frac{1}{2}}w^{c-a-b}}{\Gamma(c-a)\Gamma(a)(y+\frac{1}{2}\delta)^{\lambda+\mu+\frac{1}{2}}}$$

$$\times F_A^{(3)}\left[\lambda+\mu+\frac{1}{2},b,c-b,\mu-k+\frac{1}{2};a+b-c+1,c-a-b+1,2\mu+1;\frac{w}{y+\frac{1}{2}\delta},\frac{z}{y+\frac{1}{2}\delta},\delta\right]$$

$$+\frac{\Gamma(\lambda+\mu+c-a-b+\frac{1}{2})(\Gamma(a+b-c))^2\delta^{\mu+\frac{1}{2}}(wz)^{c-a-b}}{\Gamma(a)\Gamma(b)(y+\frac{1}{2}\delta)^{\lambda+\mu+c-a-b+\frac{1}{2}}} F_A^{(3)}\left[\lambda+\mu+c-a-b+\frac{1}{2},c-a,\right.$$

$$\left.c-b,\mu-k+\frac{1}{2};c-a-b+1,c-a-b+1,2\mu+1;\frac{w}{y+\frac{1}{2}\delta},\frac{z}{y+\frac{1}{2}\delta},\delta\right], \tag{5.2}$$

which on taking $k=\mu+\dfrac{1}{2}$ and replacing $y$ by $y-\dfrac{1}{2}\delta$, $\lambda$ by $\lambda-\mu-\dfrac{1}{2}$, reduces to a result of Olsson ([4]; p.114(3)).

Further , since integrals (1.5) and (1.6) are equivalent , taking $c=2b$ in both, then using ([7]; p.458(70)) in (1.6) and ([4]; p.114(3)) in (1.5), we get the following result:

$$\frac{\Gamma(\lambda)\,\Gamma(\frac{1}{2}+b)}{2^{1-2b}\,\Gamma(2b)} \int_0^\infty \frac{t^{2b-1}}{(t+w)^a\,(t+z)^b\,(t+y)^\lambda}\,(X)^{\frac{1}{2}-b}$$

$$\times\,(1-X)^{\frac{1}{4}(2b-2a-1)}\,P_{a-b-\frac{1}{2}}^{(\frac{1}{2}-b)}\left(\frac{2-X}{2(1-X)^{1/2}}\right)dt$$

$$=\frac{(\Gamma(b-a))^2\,\Gamma(\lambda+a-b)}{\Gamma(2b-a)\,\Gamma(b)\,(y)^{\lambda+a-b}}\,F_2\left[\lambda+a-b,\,a,\,b\,;\,a-b+1,\,a-b+1;\,\frac{w}{y},\,\frac{z}{y}\right]$$

$$+\frac{\Gamma(b-a)\,\Gamma(a-b)\,\Gamma(\lambda)}{(\Gamma(b))^2}\,\frac{z^{b-a}}{y^\lambda}\,F_2\left[\lambda,\,a,\,2b-a\,;\,a-b+1,\,b-a+1;\,\frac{w}{y},\,\frac{z}{y}\right]$$

$$+\frac{\Gamma(b-a)\,\Gamma(a-b)\,\Gamma(\lambda)}{\Gamma(a)\Gamma(2b-a)}\,\frac{w^{b-a}}{y^{\lambda}}\,F_2\left[\lambda,\,b,\,b\,;\,a-b+1,\,b-a+1;\,\frac{w}{y},\,\frac{z}{y}\right]$$

$$+\frac{(\Gamma(a-b))^2\,\Gamma(\lambda+b-a)}{\Gamma(a)\,\Gamma(b)}\,\frac{(wz)^{b-a}}{(y)^{\lambda+b-a}}$$

$$\times F_2\left[\lambda+b-a,\,2b-a,\,b\,;b-a+1,\,b-a+1;\,\frac{w}{y},\,\frac{z}{y}\right]$$

$$\left(X:=\frac{t(t+w+z)}{(t+w)(t+z)}\right)\;, \tag{5.3}$$

where $P_\nu^\mu(x)$ is the Legendre function of the first kind ([8]; p.34(29)).

Furthermore , since integrals (1.8) and (1.9) are equivalent , taking $c=2b$ in both, then using ([7]; p.458(70)) in (1.9) and (5.2) in (1.8), we get the following result:

$$\frac{\delta^{\mu+\frac{1}{2}}\Gamma(\lambda+\mu+\frac{1}{2})\Gamma(\frac{1}{2}+b)}{2^{1-2b}\,\Gamma(2b)}\int_0^\infty\frac{t^{2b-1}}{(t+w)^a\,(t+z)^b\,(t+y+\frac{1}{2}\delta)^{\lambda+\mu+\frac{1}{2}}}\,(X)^{\frac{1}{2}-b}$$

$$\times\,(1-X)^{\frac{1}{4}(2b-2a-1)}\;P_{a-b-\frac{1}{2}}^{(\frac{1}{2}-b)}\left(\frac{2-X}{2(1-X)^{1/2}}\right)$$

$$\times{}_2F_1\left[\lambda+\mu+\frac{1}{2},\,\mu-k+\frac{1}{2};\,2\mu+1;\,\frac{\delta}{t+y+\frac{1}{2}\delta}\right]dt$$

$$=\frac{\Gamma(\lambda+\mu+a-b+\frac{1}{2})(\Gamma(b-a))^2\,\delta^{\mu+\frac{1}{2}}}{\Gamma(2b-a)\Gamma(b)(y+\frac{1}{2}\delta)^{\lambda+\mu+a-b+\frac{1}{2}}}F_A^{(3)}\left[\lambda+\mu+a-b+\frac{1}{2},\right.$$

$$\left.a,b,\mu-k+\frac{1}{2};a-b+1,a-b+1,2\mu+1;\frac{w}{y+\frac{1}{2}\delta},\frac{z}{y+\frac{1}{2}\delta},\delta\right]$$

$$+\frac{\Gamma(\lambda+\mu+\frac{1}{2})\Gamma(a-b)\Gamma(b-a)\delta^{\mu+\frac{1}{2}}z^{b-a}}{(\Gamma(b))^2\;(y+\frac{1}{2}\delta)^{\lambda+\mu+\frac{1}{2}}}$$

$$\times F_A^{(3)}\left[\lambda+\mu+\frac{1}{2},a,2b-a,\mu-k+\frac{1}{2};a-b+1,b-a+1,2\mu+1;\frac{w}{y+\frac{1}{2}\delta},\frac{z}{y+\frac{1}{2}\delta},\delta\right]$$

$$+\frac{\Gamma(\lambda+\mu+\frac{1}{2})\Gamma(a-b)\Gamma(b-a)\delta^{\mu+\frac{1}{2}}w^{b-a}}{\Gamma(2b-a)\Gamma(a)(y+\frac{1}{2}\delta)^{\lambda+\mu+\frac{1}{2}}}$$

$$\times F_A^{(3)}\left[\lambda+\mu+\frac{1}{2},b,b,\mu-k+\frac{1}{2};a-b+1,b-a+1,2\mu+1;\frac{w}{y+\frac{1}{2}\delta},\frac{z}{y+\frac{1}{2}\delta},\delta\right]$$

$$+\frac{\Gamma(\lambda+\mu+b-a+\frac{1}{2})(\Gamma(a-b))^2\delta^{\mu+\frac{1}{2}}(wz)^{b-a}}{\Gamma(a)\Gamma(b)(y+\frac{1}{2}\delta)^{\lambda+\mu+b-a+\frac{1}{2}}}\,F_A^{(3)}\left[\lambda+\mu+b-a+\frac{1}{2},2b-a,\right.$$

$$b, \mu - k + \frac{1}{2}; b - a + 1, b - a + 1, 2\mu + 1; \frac{w}{y + \frac{1}{2}\delta}, \frac{z}{y + \frac{1}{2}\delta}, \delta\Bigg] \;, \tag{5.4}$$

which on taking $k = \mu + \dfrac{1}{2}$ and replacing $y$ by $y - \dfrac{1}{2}\delta$, $\lambda$ by $\lambda - \mu - \dfrac{1}{2}$, reduces to (5.3).

Once again , since integrals (1.5) and (1.6) are equivalent , taking $a = 1$ and $c = \frac{3}{2}$ in both, then using ([7]; p.462(126)) in (1.6) and ([4]; p.114(3)) in (1.5), we get the following result:

$$\frac{\Gamma(\lambda)\,\Gamma(1-b)}{\pi\,(2)^{b-\frac{3}{2}}} \int_0^\infty \frac{\sqrt{\frac{t}{X}}}{(t+w)\,(t+z)^b\,(t+y)^\lambda} (1-X)^{\frac{1}{4}(1-2b)}\, Q^{(b-\frac{1}{2})}_{\frac{1}{2}-b}(\sqrt{X})\, dt$$

$$= \frac{(\Gamma(\frac{1}{2}-b))^2\,\Gamma(\lambda+b-\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{3}{2}-b)\,(y)^{\lambda+b-\frac{1}{2}}} F_2\left[\lambda+b-\frac{1}{2},\, 1,\, b;\, b+\frac{1}{2},\, b+\frac{1}{2};\, \frac{w}{y},\, \frac{z}{y}\right]$$

$$+ \frac{\Gamma(\frac{1}{2}-b)\,\Gamma(b-\frac{1}{2})\,\Gamma(\lambda)}{\Gamma(b)\Gamma(\frac{3}{2}-b)} \frac{(z)^{\frac{1}{2}-b}}{y^\lambda} F_2\left[\lambda,\, 1,\, \frac{1}{2};\, b+\frac{1}{2},\, \frac{3}{2}-b;\, \frac{w}{y},\, \frac{z}{y}\right]$$

$$+ \frac{\Gamma(\frac{1}{2}-b)\,\Gamma(b-\frac{1}{2})\,\Gamma(\lambda)}{\sqrt{\pi}} \frac{(w)^{\frac{1}{2}-b}}{y^\lambda} F_2\left[\lambda,\, b,\, \frac{3}{2}-b;\, b+\frac{1}{2},\, \frac{3}{2}-b;\, \frac{w}{y},\, \frac{z}{y}\right]$$

$$+ \frac{(\Gamma(b-\frac{1}{2}))^2\,\Gamma(\lambda-b+\frac{1}{2})}{\Gamma(b)} \frac{(wz)^{\frac{1}{2}-b}}{(y)^{\lambda-b+\frac{1}{2}}}$$

$$\times F_2\left[\lambda-b+\frac{1}{2},\, \frac{1}{2},\, \frac{3}{2}-b; \frac{3}{2}-b,\, \frac{3}{2}-b;\, \frac{w}{y},\, \frac{z}{y}\right]$$

$$\left(X := \frac{t(t+w+z)}{(t+w)(t+z)}\right) \;, \tag{5.5}$$

where $Q^\mu_\nu(x)$ is the Legendre function of the second kind ([8]; p.35(30)).

Finally , since integrals (1.8) and (1.9) are equivalent , taking $a = 1$ and $c = \frac{3}{2}$ in both, then using ([7]; p.462(126)) in (1.9) and (5.2) in (1.8), we get the following result:

$$\frac{\delta^{\mu+\frac{1}{2}}\Gamma(\lambda+\mu+\frac{1}{2})\Gamma(1-b)}{\pi\,(2)^{b-\frac{3}{2}}} \int_0^\infty \frac{\sqrt{\frac{t}{X}}}{(t+w)\,(t+z)^b\,(t+y+\frac{1}{2}\delta)^{\lambda+\mu+\frac{1}{2}}}$$

$$\times\, (1-X)^{\frac{1}{4}(1-2b)}\, Q^{(b-\frac{1}{2})}_{\frac{1}{2}-b}(\sqrt{X})\; {}_2F_1\left[\lambda+\mu+\frac{1}{2},\, \mu - rac12;\, 2\mu+1;\, \frac{\delta}{t+y+\frac{1}{2}\delta}\right] dt$$

$$= \frac{\Gamma(\lambda+\mu+b)(\Gamma(\frac{1}{2}-b))^2\,\delta^{\mu+\frac{1}{2}}}{\sqrt{\pi}\;\Gamma(\frac{3}{2}-b)(y+\frac{1}{2}\delta)^{\lambda+\mu+b}} F_A^{(3)}\,[\lambda+\mu+b,$$

$$1, b, \mu - k + \frac{1}{2}; b + \frac{1}{2}, b + \frac{1}{2}, 2\mu + 1; \frac{w}{y + \frac{1}{2}\delta}, \frac{z}{y + \frac{1}{2}\delta}, \delta\Bigg]$$

$$+\frac{\Gamma(\lambda + \mu + \frac{1}{2})\Gamma(b - \frac{1}{2})\Gamma(\frac{1}{2} - b)\delta^{\mu + \frac{1}{2}}(z)^{\frac{1}{2} - b}}{\Gamma(\frac{3}{2} - b)\Gamma(b)\ (y + \frac{1}{2}\delta)^{\lambda + \mu + \frac{1}{2}}}$$

$$\times F_A^{(3)}\left[\lambda + \mu + \frac{1}{2}, 1, \frac{1}{2}, \mu - k + \frac{1}{2}; b + \frac{1}{2}, \frac{3}{2} - b, 2\mu + 1; \frac{w}{y + \frac{1}{2}\delta}, \frac{z}{y + \frac{1}{2}\delta}, \delta\right]$$

$$+\frac{\Gamma(\lambda + \mu + \frac{1}{2})\Gamma(b - \frac{1}{2})\Gamma(\frac{1}{2} - b)\delta^{\mu + \frac{1}{2}}(w)^{\frac{1}{2} - b}}{\sqrt{\pi}\ (y + \frac{1}{2}\delta)^{\lambda + \mu + \frac{1}{2}}}$$

$$\times F_A^{(3)}\left[\lambda + \mu + \frac{1}{2}, b, \frac{3}{2} - b, \mu - k + \frac{1}{2}; b + \frac{1}{2}, \frac{3}{2} - b, 2\mu + 1; \frac{w}{y + \frac{1}{2}\delta}, \frac{z}{y + \frac{1}{2}\delta}, \delta\right]$$

$$+\frac{\Gamma(\lambda + \mu - b + 1)(\Gamma(b - \frac{1}{2}))^2\delta^{\mu + \frac{1}{2}}(wz)^{\frac{1}{2} - b}}{\Gamma(b)(y + \frac{1}{2}\delta)^{\lambda + \mu - b + 1}} F_A^{(3)}\Bigg[\lambda + \mu - b + 1, \frac{1}{2},$$

$$\frac{3}{2} - b, \mu - k + \frac{1}{2}; \frac{3}{2} - b, \frac{3}{2} - b, 2\mu + 1; \frac{w}{y + \frac{1}{2}\delta}, \frac{z}{y + \frac{1}{2}\delta}, \delta\Bigg]$$

$$\left(X := \frac{t(t + w + z)}{(t + w)(t + z)}\right), \tag{5.6}$$

which on taking $k = \mu + \frac{1}{2}$ and replacing $y$ by $y - \frac{1}{2}\delta$ , $\lambda$ by $\lambda - \mu - \frac{1}{2}$ , reduces to (5.5).

## References


[1] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., *Tables of Integral Transforms*, Vol.I, McGraw Hill, New York, 1954.

[2] Goldstein, S., *Operational representation of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function*, Proc. Lond. Math. Soc. **34**(2)(1932), 103–125.

[3] Olsson, P.O.M., *Solutions in a special case of the partial differential equations associated with the Appell function $F_2$*, Arkiv Fysik **25**(34)(1964), 473–480.

[4] Olsson, P.O.M., *The Laplace transform of a product of two Whittaker functions*, Arkiv Fysik **28**(11)(1964), 113–120.

[5] Olsson, P.O.M., *Certain solutions of the partial differential equations associated with Appell's hypergeometric function $F_2$*, Arkiv Fysik **29**(20)(1965), 285–291.

[6] Olsson, P.O.M., *A hypergeometric function of two variables of importance in perturbation theory I*, Arkiv Fysik **30**(14)(1964), 187–191.

[7] Prudnikov, A.P., Brychkov, A. Yu. and Marichev, O.I., *Integrals and Series*, Vol. III., Gordon and Breach Science Publishers, New York, 1990.

[8] Srivastava, H.M. and Manocha, H.L., *A Treatise on Generating Functions*, Ellis Horwood Limited, Chichester, 1984.

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