Some Applications of the Result of Nunokawa to Certain Normalized Analytic Functions

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Abstract

Let \( A \) denote the class of functions \( f(z) \) which are analytic and univalent in the unit open disk \( U \) with \( f(0) = 0 \) and \( f'(0) = 1 \). By making use of the result of M. Nunokawa [Proc. Japan Acad. Ser. A Math. Sci. 68 (1992), 152-153], several result for the properties of functions \( f(z) \in A \) defined in terms of the analytic representations of convex, starlike and close-to-convex functions are proven. In addition, certain consequences of them are pointed out.

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1 Introduction and definitions

Let \( A \) denote the class of functions \( f(z) \) normalized by \( f(0) = f'(0) - 1 = 0 \) which are analytic and univalent in the unit disk \( U = \{ z : z \in C \text{ and } |z| < 1 \} \), where \( C \) is the complex plane.
A function \( f(z) \in \mathcal{A} \) is said to be starlike with respect to the origin if and only if
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathcal{U})
\]
and a function \( f(z) \in \mathcal{A} \) is said to be convex with respect to the origin if and only if
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathcal{U}).
\]
Furthermore, a function \( f(z) \in \mathcal{A} \) is said to be close-to-convex with respect to the origin if there exists a convex function \( g(z) \in \mathcal{A} \) such that
\[
\Re \left\{ \frac{f'(z)}{g'(z)} \right\} > 0 \quad (z \in \mathcal{U}).
\]
It is easily seen that the above inequality is equivalent to
\[
\Re \{ f'(z) \} > 0 \quad (z \in \mathcal{U}),
\]
when \( g(z) = z \). For the details of the above definitions, one may refer to [1] and [3].

In the present investigation, several sufficient conditions for the properties of functions \( f(z) \in \mathcal{A} \) defined in terms of the analytic representations of convex, starlike and close-to-convex functions are given. For their proofs, we used the well-known result of M. Nunokawa [2]. In addition, several special results depending on the main results are focused on.

## 2 Main Results

In order to prove one of the main results, we shall make use of the following lemma.

**Lemma 2.1** [2] Let the function \( p(z) \) be analytic in \( \mathcal{U} \) and \( p(0) = 1 \). If there exists a point \( z_0 \in \mathcal{U} \) such that
\[
\Re \{ p(z) \} > 0 \quad (|z| < |z_0|), \quad \Re \{ p(z_0) \} = 0 \quad \text{and} \quad p(z_0) \neq 0,
\]
then
\[
p(z_0) = ia \quad \text{and} \quad \frac{z_0p'(z_0)}{p(z_0)} = i \frac{c}{a} \left( a + \frac{1}{a} \right),
\]
where \( a \in \mathbb{R} \setminus \{0\} \) and \( c \geq 1 \).

By applying Lemma 2.1, we now prove next theorem.
Theorem 2.2 Let $f(z) \in A$ and $w \in C^* := C \setminus \{0\}$.

(i) If $f(z)$ satisfies

$$\left| \Re \left( \frac{zf''(z)}{f'(z)} \right) \right| < \frac{|\Im(w)|}{|w|^2} \quad \text{if} \quad \Im(w) \neq 0$$

or

$$\left| \Im \left( \frac{zf''(z)}{f'(z)} \right) \right| < \frac{|\Re(w)|}{|w|^2} \quad \text{if} \quad \Re(w) \neq 0$$

for all $z \in U$, then

$$\Re \left\{ [f'(z)]^w \right\} > 0, \quad z \in U.$$

(ii) If $f(z)$ satisfies

$$\left| \Re \left( \frac{zf''(z)}{f'(z)} \right) \right| < \frac{|\Im(w)|}{|w|^2} \quad \text{if} \quad \Im(w) \neq 0$$

or

$$\left| \Im \left( \frac{zf''(z)}{f'(z)} \right) \right| < \frac{|\Re(w)|}{|w|^2} \quad \text{if} \quad \Re(w) \neq 0$$

for all $z \in U$, then

$$\Re \left\{ [f'(z)]^{1/w} \right\} > 0, \quad z \in U.$$

In (2.3) and (2.4) powers are taken by their principal value.

Proof (i) Let us define a function $p(z)$ by

$$p(z) = [f'(z)]^w \quad (w \in C^*; z \in U).$$

Clearly, $p(z)$ is analytic in $U$ with $p(0) = 1$ and

$$zf''(z) = \frac{1}{w} \frac{zp'(z)}{p(z)}.$$

If there exists a point $z_0 \in U$ such that

$$\Re\{p(z)\} > 0 \quad (|z| < |z_0|), \quad \Re\{p(z_0)\} = 0 \quad \text{and} \quad p(z_0) \neq 0,$$

from Lemma 2.1, we have that

$$p(z_0) = ia \quad \text{and} \quad \frac{z_0p'(z_0)}{p(z_0)} = i \frac{c}{2} \left( a + \frac{1}{a} \right),$$
where \( a \in \mathcal{R} \setminus \{0\} \) and \( c \geq 1 \). Further, (2.5) and (2.6) imply
\[
\Re \left( \frac{z_0 f''(z_0)}{f'(z_0)} \right) = \Re \left( \frac{1}{w} \frac{z_0 f'(z_0)}{p(z_0)} \right) = \frac{c}{2 |w|^2} \Im m(w) \left( a + \frac{1}{a} \right)
\]
and
\[
\Im m \left( \frac{z_0 f''(z_0)}{f'(z_0)} \right) = \Im m \left( \frac{1}{w} \frac{z_0 f'(z_0)}{p(z_0)} \right) = \frac{c}{2 |w|^2} \Re(w) \left( a + \frac{1}{a} \right).
\]
Concerning that the function \( h(a) = |a + \frac{1}{a}| \) has minimal value 2, we receive
\[
\left| \Re \left( \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right| \begin{cases} \geq \frac{3m(w)}{|w|^2} & \text{if } \Im m(w) \neq 0 \\ = 0 & \text{if } \Im m(w) = 0 \end{cases}
\]
and
\[
\left| \Im m \left( \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right| \begin{cases} \geq \frac{\Re(w)}{|w|^2} & \text{if } \Re(w) \neq 0 \\ = 0 & \text{if } \Re(w) = 0 \end{cases},
\]
which is contradiction with (2.1) or (2.2). Hence, \( \Re \{p(z)\} > 0 \) for all \( z \in \mathcal{U} \), i.e., (2.3) holds.

(ii) We can easily receive inequality (2.4) if we repeat the proof of part (i) with \( 1/w \) in stead of \( w \).

By using the similar techniques as in the proof of Theorem 2.2 one can verify the results given in Theorems 2.3 and 2.4.

**Theorem 2.3** Let \( f(z) \in \mathcal{A} \) and \( w \in \mathcal{C}^* := \mathcal{C} \setminus \{0\} \).

(i) If \( f(z) \) satisfies
\[
\left| \Re \left( \frac{z f'(z)}{f(z)} \right) - 1 \right| \begin{cases} < \frac{3m(w)}{|w|^2} & \text{if } \Im m(w) \neq 0 \\ \neq 0 & \text{if } \Im m(w) = 0 \end{cases}
\]
or
\[
\left| \Im m \left( \frac{z f'(z)}{f(z)} \right) - 1 \right| \begin{cases} < \frac{\Re(w)}{|w|^2} & \text{if } \Re(w) \neq 0 \\ \neq 0 & \text{if } \Re(w) = 0 \end{cases},
\]
for all \( z \in \mathcal{U} \), then
\[
\Re \left\{ \left[ \frac{f(z)}{z} \right]^w \right\} > 0, \quad z \in \mathcal{U}.
\]

(ii) If \( f(z) \) satisfies
\[
\left| \Re \left( \frac{z f'(z)}{f(z)} \right) - 1 \right| \begin{cases} < |\Im m(w)| & \text{if } \Im m(w) \neq 0 \\ \neq 0 & \text{if } \Im m(w) = 0 \end{cases}
\]
or
\[
\left| \Im m \left( \frac{z f'(z)}{f(z)} \right) - 1 \right| \begin{cases} < |\Re(w)| & \text{if } \Re(w) \neq 0 \\ \neq 0 & \text{if } \Re(w) = 0 \end{cases},
\]
for all \( z \in \mathcal{U} \), then
\[
\Re \left\{ \left[ \frac{f(z)}{z} \right]^{1/w} \right\} > 0, \quad z \in \mathcal{U}.
\]

The powers are taken by their principal value.

**Theorem 2.4** Let \( f(z) \in \mathcal{A} \), \( w \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) and let \( \mathcal{F}(z) \) be defined by
\[
\mathcal{F}(z) = (1 - \lambda)f(z) + \lambda z f'(z) \quad (0 \leq \lambda \leq 1).
\]

(i) If \( f(z) \) satisfies
\[
\left| 1 + \Re \left\{ z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right\} \right| \begin{cases} < \frac{|\Im m(w)|}{|w|^2} & \text{if } \Im m(w) \neq 0 \\ \neq 0 & \text{if } \Im m(w) = 0 \end{cases}
\]
or
\[
\Im m \left\{ z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right\} \begin{cases} < \frac{|\Re e(w)|}{|w|^2} & \text{if } \Re e(w) \neq 0 \\ \neq 0 & \text{if } \Re e(w) = 0 \end{cases}
\]
for all \( z \in \mathcal{U} \), then
\[
\Re \left\{ \left[ \frac{z \mathcal{F}(z)}{\mathcal{F}(z)} \right]^w \right\} > 0, \quad z \in \mathcal{U}.
\]

(ii) If \( f(z) \) satisfies
\[
\left| 1 + \Re \left\{ z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right\} \right| \begin{cases} < |\Im m(w)| & \text{if } \Im m(w) \neq 0 \\ \neq 0 & \text{if } \Im m(w) = 0 \end{cases}
\]
or
\[
\Im m \left\{ z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right\} \begin{cases} < |\Re e(w)| & \text{if } \Re e(w) \neq 0 \\ \neq 0 & \text{if } \Re e(w) = 0 \end{cases}
\]
for all \( z \in \mathcal{U} \), then
\[
\Re \left\{ \left[ \frac{z \mathcal{F}'(z)}{\mathcal{F}(z)} \right]^{1/w} \right\} > 0, \quad z \in \mathcal{U}.
\]

The powers are taken by their principal value.

By putting \( w := 1 \) in Theorem 2.2 and Theorem 2.3, respectively, we find that:

**Corollary 2.5** Let \( f(z) \in \mathcal{A} \) and also let
\[
\Re \left( \frac{z f''(z)}{f'(z)} \right) \neq 0 \quad \text{or} \quad \Im m \left( \frac{z f''(z)}{f'(z)} \right) < 1
\]
for all \( z \in \mathcal{U} \). Then \( \Re \{ f'(z) \} > 0 \), \( z \in \mathcal{U} \), i.e., \( f(z) \) is a close-to-convex function.
Corollary 2.6 Let $f(z) \in \mathcal{A}$ and also let
\[ \Re \left( \frac{zf'(z)}{f(z)} \right) \neq 1 \quad \text{or} \quad \Re \left( \frac{zf'(z)}{f(z)} - 1 \right) < 1 \]
for all $z \in \mathcal{U}$. Then $\Re \left\{ \frac{f(z)}{z} \right\} > 0$.

By taking $w := 1$ and $\lambda := 0$ in Theorem 2.4, we get that:

Corollary 2.7 Let $f(z) \in \mathcal{A}$ and also let
\[ \Re \left\{ z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right\} \neq -1 \]
or
\[ \Re \left\{ z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right\} < 1 \]
for all $z \in \mathcal{U}$. Then $\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$, $z \in \mathcal{U}$, i.e., $f(z)$ is a starlike function.

By taking $w := 1$ and $\lambda := 1$ in Theorem 2.4, we also get that:

Corollary 2.8 Let $f(z) \in \mathcal{A}$ and also let
\[ \Re \left\{ z \left( \frac{(zf'(z))^n}{(zf'(z))^m} - \frac{(zf'(z))^m}{zf'(z)} \right) \right\} \neq -1 \]
or
\[ \Re \left\{ z \left( \frac{(zf'(z))^n}{(zf'(z))^m} - \frac{(zf'(z))^m}{zf'(z)} \right) \right\} < 1 \]
for all $z \in \mathcal{U}$. Then $\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > -1$, $z \in \mathcal{U}$, i.e., $f(z)$ is a convex function.

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References


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