Int. Journal of Math. Analysis, Vol. 1, 2007, no. 8, 369 - 378

On Entire Functions of Exponential Type with an Infinite Number of Sign Changes

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Abstract. We study a class of entire functions of exponential type that are real on the real axis and that change signs an infinite number of times.

Mathematics Subject Classification: 30D15, 46F10

Keywords: Exponential type, tempered distributions

1. Introduction

Several classes of entire functions of exponential type have been studied extensively [1], especially because of their importance in signal processing [11]. One of the interesting features of functions of these classes is their oscillatory properties. The aim of this article is to study some questions on the number of sign changes, and of zeros, in particular, for entire functions whose restriction to the real axis is a tempered distribution.

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a function that is also a tempered distribution, $f \in \mathcal{S}'(\mathbb{R})$. Let $g = \widehat{f} = \mathcal{F}(f)$ be its Fourier transform. The spectrum of f is the support of g, $\operatorname{supp} g$; if this set is contained in an interval of the type $[-\tau, \tau]$ then f can be extended to $\mathbb{C}$ as an entire function of exponential type τ. We use the notation $f \in \mathfrak{G}_\tau$, and call f a distributional Paley-Wiener function of type τ. When $f \in \mathfrak{G}_\tau$ is bounded on the real axis we use the notation $f \in \mathfrak{B}_\tau$; the class $\mathfrak{B}_\tau$ was introduced in [2], where several properties related to oscillation and the location of zeros were considered. If $f \in \mathfrak{G}_\tau$ and $f \in L^2(\mathbb{R})$, or, what is the same, if $g \in L^2(\mathbb{R})$, then we call f an (ordinary) Paley-Wiener function of type τ, and use the notation $f \in \mathfrak{P}_\tau$; the class $\mathfrak{P}_\tau$ plays an important role in the well-known Shannon sampling theorem [11], but as Walter has shown

[19, 20], the sampling theorem remains valid in the summability sense if $f \in \mathfrak{G}_\tau$ whenever g is a strongly integrable distribution on $[-\tau, \tau]$.

There are positive Paley-Wiener functions [10]. However, a Paley-Wiener function may have an infinite number of zeros if strict smooth conditions are imposed on its Fourier transform [18]. In [10] Higgins asked whether any real valued Paley-Wiener function $f \in \mathfrak{P}_\tau$ with spectrum contained in a set of the form $[-\tau, -\sigma] \cup [\sigma, \tau]$ for some $\sigma \in (0, \tau]$ should have an infinite number of zeros. A positive answer was obtained in [2], where it is proved that if a real valued function $f \in \mathfrak{B}_\tau$ has spectrum contained in $[-\tau, -\sigma] \cup [\sigma, \tau]$ for some $\sigma \in (0, \tau]$, then it has an infinite number of real zeros. In Section 3 we give a generalization of this result; in fact we prove that a real valued function f has an infinite number of sign changes, in particular, an infinite number of zeros, if $f \in \mathfrak{G}_\tau$, even if the spectrum of f is not contained in any set of the form $[-\tau, -\sigma] \cup [\sigma, \tau]$, as long as the Fourier transform $g = \widehat{f}$ vanishes of infinite order at the origin *in the distributional sense,* that is,

$$g^{(k)}(0) = 0\,, \quad \text{distributionally}, \quad \forall k \in \mathbb{N}\,. \tag{1.1}$$

We explain the idea of distributional values in Section 2. The result holds of course if $g^{(k)}(0) = 0$ $\forall k \in \mathbb{N}$ in the ordinary sense.

Actually we prove in Theorem 1 that a real valued continuous function $f \in \mathcal{S}'(\mathbb{R})$ whose Fourier transform satisfies (1.1) changes sign an infinite number of times. If we define the idea of *sign change* properly, there is no need to assume that f is continuous; we prove that if f is a Radon signed measure and (1.1) holds then f changes signs an infinite number of times. We apply these ideas to Dirac combs, and prove that if $f \in \mathfrak{G}_\tau$ then the sequence $\{f(n\xi)\}_{n=-\infty}^{\infty}$ changes sign an infinite number of times if ξ is small, $0 < \xi < 2\pi/\tau$. We also show that the sequence $\{f(2\pi n/\tau)\}_{n=-\infty}^{\infty}$ might not change signs, in general, but give conditions that guarantee that it does.

Another question raised by Higgins in [10] is whether the derivatives of Paley-Wiener functions become more oscillatory, in the following sense, whether for each real valued $f \in \mathfrak{P}_\tau$ there exists k, which depends on f, such that $f^{(k)}$ has infinite zeros. This question was answered in the negative in [14, 15], where one can find examples of real valued Paley-Wiener functions whose derivatives of any order have only a finite number of zeros; they also studied several properties of such functions. In Section 4 we give yet another constructions of such "non-oscillating" real valued Paley-Wiener functions as certain Hilbert transforms.

Naturally, there are functions whose derivatives are never zero, as $f(x) = e^{\alpha x}$, $\alpha \in \mathbb{R} \setminus \{0\}$. These functions were considered in the classic work of Bernstein (see [3, Section 10]). On the other hand, it is clear that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ vanishes at $\pm\infty$, then the number of zeros of $f^{(k)}$, say $\nu_k(f)$, tend to ∞ as k increases. In Section 4 we prove the curious result that if f is smooth and

$f \in \mathcal{S}'(\mathbb{R})$ then $\lim_{k\to\infty} \nu_k(f) = \infty$. Observe that there are smooth elements of $\mathcal{S}'(\mathbb{R})$ whose derivatives do not vanish at $\pm\infty$, and that actually are of exponential order at $\pm\infty$.

2. Preliminaries

In this section we explain some concepts that will be needed in the rest of the article, especially the notion of Cesàro behavior of a distribution at infinity [7] and at a point [9, 13].

We will work on the usual distribution spaces $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$. We choose the constants in our definition of the Fourier transform in such a way that

$$\widehat{f}(u) = \mathcal{F}\{f(x); u\} = \int_{-\infty}^{\infty} f(x)\, e^{ixu}\, \mathrm{d}x\,, \tag{2.1}$$

whenever the integral converges, and define $\mathcal{F}$ by the usual duality process if $f \in \mathcal{S}'(\mathbb{R})$ [12].

The Cesàro behavior of a distribution at infinity is studied by using the order symbols $O(x^\alpha)$ and $o(x^\alpha)$ in the Cesàro sense. If $f \in \mathcal{D}'(\mathbb{R})$ and $\alpha \in \mathbb{R}\setminus\{-1,-2,-3,\dots\}$, we say that $f(x) = O(x^\alpha)$ as $x \to \infty$ in the Cesàro sense and write $f(x) = O(x^\alpha)$ (C), as $x \to \infty$, if there exists $N \in \mathbb{N}$ such that every primitive F of order N of f, i.e., $F^{(N)} = f$, is an ordinary function for large arguments and satisfies the ordinary order relation $F(x) = p(x) + O(x^{\alpha+N})$, as $x \to \infty$, for a suitable polynomial p of degree $N-1$ at the most. A similar definition applies to the little o symbol. The definitions when $x \to -\infty$ are clear. One can also consider the case when $\alpha = -1,-2,-3,\dots$ [9, Def. 6.3.1].

These ideas can be readily extended to the study of the local behavior of generalized functions [9]. Actually, Łojasiewicz [13] defined the value of distribution $f \in \mathcal{D}'(\mathbb{R})$ at the point x_0 as the limit $f(x_0) = \lim_{\varepsilon\to 0} f(x_0+\varepsilon x)$, if the limit exists in $\mathcal{D}'(\mathbb{R})$, that is, if for each $\phi \in \mathcal{D}(\mathbb{R})$, $\lim_{\varepsilon\to 0}\langle f(x_0+\varepsilon x), \phi(x)\rangle = f(x_0)\int_{-\infty}^{\infty}\phi(x)\,\mathrm{d}x$. For example the generalized function $f(x) = \sin(1/x)$ is oscillatory near $x = 0$, however, it is easy to see that $f(0)$ exists and equals 0.

Let $f \in \mathcal{D}'(\mathbb{R})$ with support bounded on the left. If $\phi \in \mathcal{E}(\mathbb{R})$ then the evaluation $\langle f(x), \phi(x)\rangle$ will not be defined, in general. We say that the evaluation exists in the Cesàro sense and equals L, written as

$$\langle f(x), \phi(x)\rangle = L \quad (\mathrm{C})\,, \tag{2.2}$$

if $g(x) = L + o(1)$ (C) as $x \to \infty$, where g is the primitive of $f\phi$ with support bounded on the left. A similar definition applies if $\operatorname{supp} f$ is bounded on the right. Observe that if f is locally integrable with $\operatorname{supp} f \subset [a,\infty)$ then (2.2) means that $\int_a^\infty f(x)\,\phi(x)\;\mathrm{d}x = L$ (C), while if $f(x) = \sum_{n=0}^\infty a_n\delta(x-n)$ then (2.2) tells us that $\sum_{n=0}^\infty a_n\phi(n) = L$ (C).

In the general case when the support of f extends to both $-\infty$ and $+\infty$, there are various different but related notions of evaluations in the Cesàro sense (or in any other summability sense, in fact). If f admits a representation of the form $f = f_1 + f_2$, with $\operatorname{supp} f_1$ bounded on the left and $\operatorname{supp} f_2$ bounded on the right, such that $\langle f_j(x), \phi(x)\rangle = L_j$ (C) exist, then we say that the (C) evaluation $\langle f(x), \phi(x)\rangle$ (C) exists and equals $L = L_1 + L_2$. This is clearly independent of the decomposition. The notation (2.2) is used in this situation. It happens many times that $\langle f(x), \phi(x)\rangle$ (C) does not exist, but the symmetric limit, $\lim_{x\to\infty}\{g(x) - g(-x)\} = L$, where g is any primitive of $f\phi$, exists in the (C) sense. Then we say that the evaluation $\langle f(x), \phi(x)\rangle$ exists in the principal value Cesàro sense, and write p.v. $\langle f(x), \phi(x)\rangle = L$ (C). An intermediate notion, very useful for our purposes, is the following. If

$$\lim_{x\to\infty}\{g(ax) - g(-x)\} = L \quad (\mathrm{C})\,, \quad \forall a > 0\,, \tag{2.3}$$

we say that the distributional evaluation exists in the e.v. Cesàro sense and write

$$\text{e.v.}\,\langle f(x), \phi(x)\rangle = L \quad (\mathrm{C})\,. \tag{2.4}$$

The characterization of the Fourier series of those periodic distributions that have a distributional point value was given in [6]: if $f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ in the space $\mathcal{D}'(\mathbb{R})$ then $f(\theta_0) = \gamma$, distributionally, if and only if

$$\lim_{x\to\infty}\sum_{-x\le n\le ax} a_n e^{in\theta_0} = \gamma \quad (\mathrm{C})\,, \quad \forall a > 0\,. \tag{2.5}$$

This characterization was extended to general tempered distributions in [16]. Let $f \in \mathcal{S}'(\mathbb{R})$. If $x_0 \in \mathbb{R}$ then $f(x_0) = \gamma$, distributionally, if and only if

$$\text{e.v.}\left\langle \widehat{f}(u), e^{-iux_0}\right\rangle = 2\pi\gamma \quad (\mathrm{C})\,, \tag{2.6}$$

which in case $\widehat{f}$ is locally integrable means that

$$\text{e.v.}\int_{-\infty}^{\infty}\widehat{f}(u)\, e^{-iux_0}\mathrm{d}u = 2\pi\gamma \quad (\mathrm{C})\,. \tag{2.7}$$

3. Functions with sign changes

Our first result is the following.

Theorem 1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, $f \neq 0$. Suppose that $f \in \mathcal{S}'(\mathbb{R})$ and let $g = \widehat{f}$. If*

$$g^{(k)}(0) = 0\,, \quad \textit{distributionally,} \quad \forall k \in \mathbb{N}\,, \tag{3.1}$$

then f changes sign an infinite number of times.

Proof. Since $g^{(k)}(x) = i^k\mathcal{F}\left\{t^k f(t); x\right\}$, (3.1) combined with (2.7) yields that

$$\text{e.v.}\int_{-\infty}^{\infty} t^k f(t)\,\mathrm{d}t = 0 \quad (\mathrm{C}) \quad \forall k \in \mathbb{N}\,. \tag{3.2}$$

It follows that e.v. $\int_{-\infty}^{\infty} p(t) f(t) \, dt = 0$ (C) for each polynomial p.

Suppose f changes sign only a finite number of times. Let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be the points where f changes sign and which are not the left endpoint of an interval where f vanishes. Let $p_0(t) = (t - \alpha_1) \cdots (t - \alpha_N)$. Then $p_0(t) f(t)$ never changes sign. But if e.v. $\int_{-\infty}^{\infty} h(t) \, dt = \gamma$ (C) and $h(t) \geq 0 \ \forall t$ then e.v. $\int_{-\infty}^{\infty} h(t) \, dt = \gamma$. Since e.v. $\int_{-\infty}^{\infty} p_0(t) f(t) \, dt = 0$ (C) it follows that e.v. $\int_{-\infty}^{\infty} p_0(t) f(t) \, dt = 0$, and thus $p_0(t) f(t) = 0 \ \forall t \in \mathbb{R}$, and because f is continuous, $f = 0$, a contradiction. □

The proof of Theorem 1 actually shows that if $g^{(k)}(0) = 0$, distributionally, for $0 \leq k \leq N$, then f has at least $N+1$ sign changes. If f is complex valued and (3.1) holds, then both $\Re e\, f$ and $\Im m\, f$ will have an infinite number of sign changes, but not at the same points, in general. The Theorem applies, in particular, to the case when $g = \widehat{f}$ has support in a set of the form $[-\tau, -\sigma] \cup [\sigma, \tau]$, therefore we obtain the result of [2]. Nevertheless, we do not need to require f to be bounded, just that $f \in \mathcal{S}'(\mathbb{R})$. Also, there is no need to assume that g has compact support, nor there is need to assume that $0 \notin \operatorname{supp} g$. The theorem applies if, for instance, $g(x) \sim |x|^{\gamma} e^{-1/|x|^{\beta}}$, or $g(x) \sim \operatorname{sgn} x \, |x|^{\gamma} e^{-1/|x|^{\beta}}$, for $\beta > 0$, which produce *ordinary* zeros of infinite order at $x = 0$, and also if $g(x) \sim |x|^{\gamma} \sin |x|^{-\beta}$, or $g(x) \sim |x|^{\gamma} \operatorname{sgn} x \, \sin(|x|^{-\beta})$, $(\beta > 0)$, or similar combinations that yield *distributional* zeros of infinite order at the origin.

We would like to emphasize the case of functions of exponential type.

Corollary 1. *Let f be an entire function of exponential type that is real on the real axis. If its restriction to the real axis is a tempered distribution with Fourier transform g, and $g^{(k)}(0) = 0$, distributionally, $\forall k \in \mathbb{N}$, then f changes sign infinite times on the real axis.*

The Theorem 1 remains valid if f is not necessarily continuous, but a general tempered distribution. For our purposes we will consider the case when f is a Radon signed measure. We need the following definition.

Definition. Let $f \in \mathcal{S}'(\mathbb{R})$ be a Radon signed measure. We say that f changes sign at the point t_0 if $t_0 \in \operatorname{supp} f$ and if there are non-empty intervals $I_1 = (a, t_0)$, $I_2 = [t_0, b)$ or $I_1 = (a, t_0]$, $I_2 = (t_0, b)$ such that the restrictions $f|_{I_1}$ and $f|_{I_2}$ are non-null Radon measures of constant and opposite signs.

If f changes sign at $t = t_1$ and at $t = t_2$, and $f(t) = 0$ for $t_1 \leq t \leq t_2$, then one can say that f changes sign once, not twice, as in the counting function of the number of sign changes used in [4].

Observe that if f is a Radon signed measure of the form

$$f(t) = \sum_{n=-\infty}^{\infty} a_n \delta(t - t_n) , \tag{3.3}$$

where $a_n \neq 0$ $\forall n$, and where $t_n \to \pm\infty$ as $n \to \pm\infty$, then f changes sign at $t = t^*$ if and only if $t^* = t_n$ for some n and $a_{n-1}a_n < 0$ or $a_n a_{n+1} < 0$, or both.

The following result extends Theorem 1 to Radon signed measures.

Theorem 2. *Let f be a Radon signed measure, $f \neq 0$. Suppose that $f \in \mathcal{S}'(\mathbb{R})$ and let $g = \widehat{f}$. If $g^{(k)}(0) = 0$, distributionally, $\forall k \in \mathbb{N}$, then f changes sign an infinite number of times.*

Proof. If f changes sign a finite number of times, then there exists a polynomial p such that pf does not change signs at all. The same argument used in the proof of Theorem 1 applies to conclude that $pf = 0$, and thus that f is a sum of Dirac delta functions concentrated on a finite set. Hence $f = 0$. □

We now apply the Theorem 2 to obtain that if $f \in \mathfrak{G}_\tau$ and (3.1) is satisfied, then sequences of the form $\{f(n\xi)\}_{n=-\infty}^{\infty}$ will change signs an infinite number of times if ξ is small enough.

Theorem 3. *Let f be an entire function of exponential type of the class $\mathfrak{G}_\tau$. Suppose f is real on the real axis and $g = \widehat{f}$ satisfies that $g^{(k)}(0) = 0$, distributionally, $\forall k \in \mathbb{N}$. If $0 < \xi < 2\pi/\tau$, then the sequence $\{f(n\xi)\}_{n=-\infty}^{\infty}$ changes sign an infinite number of times.*

Proof. Let $\sigma = 2\pi/\xi$, so that $\sigma > \tau$. Let $C_\sigma(x) = \sum_{n=-\infty}^{\infty} \delta(x - n\sigma)$ be the Dirac comb of mesh σ. Let $f = \widehat{h}$ where $h^{(k)}(0) = 0$, distributionally, $\forall k \in \mathbb{N}$, and let $h_\sigma = h * C_\sigma$. Since $\operatorname{supp} h \subseteq [-\tau, \tau]$ and $\sigma > \tau$, it follows that $h_\sigma^{(k)}(0) = 0$, distributionally, $\forall k \in \mathbb{N}$. Thus $\widehat{h_\sigma}$ changes sign an infinite number of times. But $\widehat{h_\sigma}(t) = \widehat{h}(t)\,\widehat{C_\sigma}(t)$, so

$$\widehat{h_\sigma}(t) = f(t)\,\xi \sum_{n=-\infty}^{\infty} \delta(x - n\xi) = \xi \sum_{n=-\infty}^{\infty} f(n\xi)\,\delta(x - n\xi) ,$$

and the result follows. □

Observe that if the sequence $\{a_n\}_{n=-\infty}^{\infty}$ changes sign an infinite number of times then the two sets $\{n \in \mathbb{N} : a_n > 0\}$ and $\{n \in \mathbb{N} : a_n < 0\}$ or the two sets $\{n \in \mathbb{N} : a_{-n} > 0\}$ and $\{n \in \mathbb{N} : a_{-n} < 0\}$ are both infinite.

The result of the Theorem 3 does not hold if $\xi = 2\pi/\tau$. For instance, if $f(t) = \cos \tau t$, then $f(n(2\pi/\tau)) = 1$ $\forall n \in \mathbb{Z}$, even though $\widehat{f}(x) = \pi(\delta(x - \tau) + \delta(x + \tau))$ vanishes in a neighborhood of $x = 0$. However, we have the ensuing theorem.

Theorem 4. *Let f be an entire function of exponential type of the class $\mathfrak{G}_\tau$. Suppose f is real on the real axis. If $g = \widehat{f}$ satisfies*

$$g^{(k)}(0) = g^{(k)}(\tau) = g^{(k)}(-\tau) = 0\,, \quad \textit{distributionally}\,, \ \forall k \in \mathbb{N}\,, \tag{3.4}$$

then the sequence $\{f(2\pi n/\tau)\}_{n=-\infty}^{\infty}$ changes sign an infinite number of times.

Proof. Indeed, if (3.4) holds, then h_τ defined in the proof of the previous theorem satisfies $h_\tau^{(k)}(0) = 0$, distributionally, $\forall k \in \mathbb{N}$. □

The Theorems 3 and 4 complement the results on the location of the zeros of functions of exponential type. It is known that the maximum separation of consecutive zeros, $M(f)$, satisfies $M(f) > \pi/\tau$ if $f \in \mathfrak{P}_\tau$ [17], while $M(f) \geq \pi/\tau$ when $f \in \mathfrak{G}_\tau$ [5]; when $f \in \mathfrak{B}_\tau$ then $M(f) > \pi/\tau$, unless $f = c\cos(\tau x + \alpha)$, in which case $M(f) = \pi/\tau$ [2].

4. Other results

In [14, 15] one can find examples of real valued Paley-Wiener functions whose derivatives of any order have only a finite number of zeros. The following result gives more examples of such functions.

Theorem 5. *Let $f \in \mathfrak{P}_\tau$ be a real valued function whose Fourier transform $g = \widehat{f}$ is smooth in $\mathbb{R}$ and satisfies $g(0) \neq 0$. Let*

$$F(t) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{f(s)}{s-t}\,\mathrm{d}s\,, \tag{4.1}$$

be its Hilbert transform, where the principal value is taken at $s = t$. Then $F \in \mathfrak{P}_\tau$ and $\forall k \in \mathbb{N}$ the function $F^{(k)}$ has only a finite number of real zeros.

Proof. It is clear that $F \in \mathfrak{P}_\tau$. Using [8, Lemma 6.3.1] it follows that $F(t)$ has the *strong* asymptotic expansion

$$F(t) \sim -\frac{1}{\pi}\left(\frac{\mu_0}{t} + \frac{\mu_1}{t^2} + \frac{\mu_2}{t^3} + \cdots\right)\,, \tag{4.2}$$

as $|t| \to \infty$, where $\mu_n = \int_{-\infty}^{\infty} f(s)\, s^n\, \mathrm{d}s$ are the moments. Strong means that the asymptotic formula can be differentiated any number of times. Since $\mu_0 = g(0) \neq 0$, we obtain that

$$\lim_{|t|\to\infty} t^{k+1} F^{(k)}(t) = \frac{(-1)^{k+1}}{\pi} k!\mu_0 \neq 0\,,$$

and it immediatly follows that $F^{(k)}$ has only a finite number of real zeros. □

It is interesting to observe that in case $g = \widehat{f}$ satisfies (3.1), then not only f but also F, its Hilbert transform, changes sign an infinite number of times because $\widehat{F}(x) = i\,\mathrm{sgn}\,x\, g(x)$ also satisfies that

$$\widehat{F}^{(k)}(0) = 0\,, \quad \text{distributionally}\,, \quad \forall k \in \mathbb{N}\,. \tag{4.3}$$

Notice also that if $g^{(k)}(0) = 0$, distributionally, $\forall k \in \mathbb{N}$ then the Hilbert transform F is well-defined [8, Chp. 6].

Theorem 6. *Let $f \in \mathfrak{G}_\tau$ be a real function such that its Fourier transform $g = \widehat{f}$ vanishes of infinite order at the origin in the distributional sense. Let F be its Hilbert transform, given by (4.1). Then F changes sign an infinite number of times.*

While it is true that there are real valued Paley-Wiener functions such that no derivative has infinite zeros, one may ask if $\nu_k(f)$, the number of zeros of the derivative $f^{(k)}$, becomes large as $k \to \infty$. There are functions f for which $\nu_k(f) = 0$ for each k, such as $f(t) = e^t$, or as Bernstein proved, the Laplace transforms of positive measures, which are basically all the functions with this property [3, Section 10]. If $f(x) = te^t$, then $\nu_k(f) = 1$ for all k. If, on the other hand, $f(t)$ vanishes as $|t| \to \infty$, it is obvious that $\nu_k(f) \to \infty$ as $k \to \infty$. Therefore the Theorem 7 is interesting, because the functions of the space $\mathcal{S}'(\mathbb{R})$ do not need to vanish at $\pm\infty$, they can actually show exponential growth at $\pm\infty$.

We start with some basic lemmas.

Lemma 1. *Let $g \in \mathcal{D}'(\mathbb{R})$. There exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ then $h_k(x) = x^k g(x)$ satisfies*

$$h_k(0) = 0, \quad \text{distributionally.} \tag{4.4}$$

Proof. If $G \in \mathcal{D}'(\mathbb{R})$ is continuous in a neighborhood of $x = 0$, then $\langle G(\varepsilon x), \phi(x)\rangle = O(1)$ as $\varepsilon \to 0$ for each $\phi \in \mathcal{D}(\mathbb{R})$. Hence $G^{(n)}(\varepsilon x) = O(\varepsilon^{-n})$ weakly in $\mathcal{D}'(\mathbb{R})$.

Since any distribution of $\mathcal{D}'(\mathbb{R})$ is locally of finite order, it follows that there exists $n \in \mathbb{N}$ and $G \in \mathcal{D}'(\mathbb{R})$ continuous in a neighborhood of $x = 0$, such that $G^{(n)} = g$. Then we may take $k_0 = n + 1$. This yields $h_k(\varepsilon x) = o(1)$ weakly as $\varepsilon \to 0$ for $k \geq k_0$, and (4.4) follows. □

Lemma 2. *Let $g \in \mathcal{D}'(\mathbb{R})$. Suppose $g(0) = 0$ distributionally. If $n \in \mathbb{N}$, let $q_n(x) = x^n g^{(n)}(x)$, and let $p_n(x) = (xg(x))^{(n)}$. Then $q_n(0) = 0$ distributionally and $p_n(0) = 0$ distributionally.*

Proof. Indeed, if $\phi \in \mathcal{D}(\mathbb{R})$, then

$$\begin{aligned} \langle q_n(\varepsilon x), \phi(x)\rangle &= \varepsilon^n \left\langle g^{(n)}(\varepsilon x), x^n \phi(x)\right\rangle \\ &= \left\langle g(\varepsilon x), (x^n \phi(x))^{(n)}\right\rangle \\ &= o(1), \quad \text{as }, \ \varepsilon \to 0. \end{aligned}$$

Hence $q_n(0) = 0$ distributionally.

If we now apply the Leibnitz rule we obtain that

$$p_n(x) = \sum_{j=0}^{n} \binom{n}{j} q_j(x),$$

and thus $p_n(0) = 0$ distributionally. □

We can now give the following theorem.

Theorem 7. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function. Suppose that $f \in \mathcal{S}'(\mathbb{R})$. Let $\nu_k(f)$ be the number of zeros of the derivative $f^{(k)}$. Then $\lim_{k\to\infty} \nu_k(f) = \infty$.*

Proof. Let $g = \widehat{f}$. Then $g_k(x) = (ix)^k g(x)$ is the Fourier transform of $f^{(k)}$. Let $N \in \mathbb{N}$ be fixed. There exists k such that $g_k(0) = 0$, distributionally. Hence $g_{k+q}^{(q)}(0) = 0$, distributionally $\forall q$, and consequently, $g_{k+N}^{(j)}(0) = 0$, distributionally, for $0 \leq j \leq N$. Therefore, $f^{(k+N)}$ has at least $N+1$ zeros. Thus, $\nu_{k+N}(f) \geq N+1$, and we obtain that $\lim_{k\to\infty} \nu_k(f) = \infty$. □

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Received: October 26, 2006