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Quasiconformal Mappings and the Max Property[1]

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Abstract

Suppose that D is a Jordan proper subdomain of $\overline{R}^2$ whose boundary contains at least three points, $D^* = \overline{R}^2 \setminus \overline{D}$, the exterior of D. We say that D has the max property if there exists a family Γ of curves in D and a constant $c > 1$ such that each pair of points $x_1, x_2 \in D \setminus \{\infty\}$ can be joined by a curve $\gamma \in \Gamma$ and such that

$$|x - y| \leq c \max_{j=1,2} |x_j - y|$$

for each ordered triple of points $x_1, x, x_2 \in \gamma$ and each $y \in \partial D \setminus \{\infty\}$.

In this paper, the author proves the following two results: (1) D is a quasidisk if and only if both D and D^* have the max property; (2) A homeomorphism $f : \overline{R}^2 \to \overline{R}^2$ with $f(\infty) = \infty$ is a quasiconformal mapping if and only if $f(D)$ has the max property for any $D \subseteq \overline{R}^2$ with the max property.

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1 Introduction

We shall assume throughout this paper that D is a Jordan proper subdomain of $\overline{R}^2$ whose boundary contains at least three points, $D^* = \overline{R}^2 \setminus \overline{D}$, the exterior

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of D. For convenience we shall adopt the notation and terminology as in paper [18]. For $x \in R^2$ and $0 < r < \infty$, let $B^2(x,r) = \{z \in R^2 : \ |z-x| < r\}$, $\overline{B}^2(x,r)$ be the closure of $B^2(x,r)$, $S^1(x,r) = \partial B^2(x,r)$, $B^2(r) = B^2(0,r)$ and $B^2 = B^2(1)$. Suppose that f is a homeomorphism in $\overline{R}^2$ with $f(\infty) = \infty$, for $x \in R^2$ and $0 < r < \infty$, let $L(x,f,r) = \max\limits_{|y-x|=r} |f(y) - f(x)|$, and $l(x,f,r) = \min\limits_{|y-x|=r} |f(y) - f(x)|$.

Suppose that $c > 1$ is a constant, we say that D has the $c-$max property if there exists a family Γ of curves in D such that each pair of points $x_1, x_2 \in D \setminus \{\infty\}$ can be joined by a curve $\gamma \in \Gamma$ and such that for each $\gamma \in \Gamma$,

$$|x-y| \le c \max_{j=1,2} |x_j - y| \tag{1.1}$$

for each ordered triple of points $x_1, x, x_2 \in \gamma$ and each $y \in \partial D \setminus \{\infty\}$. And we say that D has the max property if D has the $c-$max property for some $c > 1$.

The max property is an important concept in analysis and geometry, it was first introduced by F.W.Gehring and K. Hag in their study [5] of uniform and quasiconformal extension domains, in which they proved that a proper domain E of R^n is a uniform domain if and only if E has the max-min property. Later, it has been used extensively in the research of John disks by O.J. Broch[2], Z. Balogh and A. Volberg[3], R. Näkki and J. Väisälä[15], K. Hag and P. Hag[9], and K. Kim[11].

D is called a quasidisk if there exists a $K-$quasiconformal mapping $(K \ge 1)$ $f: \overline{R}^2 \to \overline{R}^2$ such that D is the image of the unit disk B^2 under f.

It is well-known that quasidisks play a very important role in quasiregular mappings[10], Kleinian groups[12], complex dynamics[13,16] and Teichmüller space theory[1,7,14,19], etc.

The purpose of this paper is to prove the following two results.

Theorem 1.1. *D is a quasidisk if and only if both D and D^* have the max property.*

Theorem 1.2. *A homeomorphism $f : \overline{R}^2 \to \overline{R}^2$ with $f(\infty) = \infty$ is a quasiconformal mapping if and only if $f(D)$ has the max property for any $D \subseteq \overline{R}^2$ with the max property.*

For the sake of convenience we shall introduce the following important concepts, they will be used to prove our theorem 1.1 and theorem 1.2 in the next sections.

Let $c \ge 1$ be a constant. (1) If for any $x_0 \in R^2$ and $0 < r < +\infty$, each pair of points $x_1, x_2 \in D \cap \overline{B}^2(x_0, r)$ can be joined by a curve in $D \cap \overline{B}^2(x_0, cr)$, then we say that D is a $c-$inner linearly locally connected domain, denoted by $D \in c-ILC$; (2) If for any $x_0 \in R^2$ and $0 < r < +\infty$, each pair of finite points $x_1, x_2 \in D \setminus B^2(x_0, r)$ can be joined by a curve in $D \setminus B^2(x_0, r/c)$, then we say that D is a $c-$outer linearly locally connected domain, denoted by $D \in c - OLC$.

D is called a linearly locally connected domain if $D \in c - ILC$ and $D \in c - OLC$ at the same time for some $c \geq 1$.

The following example shows that there exists a domain D which is not $c - ILC$ and $c - OLC$ at the same time for any $c \geq 1$.

Example.

Let

$$\begin{array}{rcl} D_1 & = & \{(x_1, x_2) | x_1^2 + (x_2 - 1)^2 < 1, x_1 < 0, x_2 < 1\}, \\ D_2 & = & \{(x_1, x_2) | x_1^2 + (x_2 + 1)^2 < 1, x_1 < 0, x_2 > -1\}, \\ D_3 & = & \{(x_1, x_2) | x_1 \geq 0, -1 < x_2 < 1\} \end{array}$$

and

$$D = (D_1 \cup D_2 \cup D_3) \setminus \{(0, 0)\}.$$

Then the simple connected domain $D \subseteq R^2$ is not $c - ILC$ and $c - OLC$ at the same time for any $c \geq 1$. In fact,

(I) For $x > 0$, denote A the point $(-x, 0)$, O the point $(0, 0)$, $d(A, \partial D)$ and $d(A, O)$ the Euclidean distance from A to ∂D and O, respectively. Then

$$\lim_{x \to 0} \frac{d(A, O)}{d(A, \partial D)} = \lim_{x \to 0} \frac{x}{\sqrt{1 + x^2} - 1} = +\infty.$$

Hence for any $c \geq 1$, there exists a point $A(-x, 0)$ such that $d(A, O) > 4cd(A, \partial D)$. This concludes that there exist points in $D \cap \overline{B}^2(A, 2d(A, \partial D))$ which can not be joined by curves in $D \cap \overline{B}^2(A, 2cd(A, \partial D))$, thus D is not $c - ILC$.

(II) For any $c \geq 1$, denote B the point $(3c^2, 0)$, $r = 2c^2$. It is obvious that there exist points in $D \setminus B^2(B, r)$ which can not be joined by curves in $D \setminus B^2(B, r/c)$, thus D is not $c - OLC$.

Let $b > 0$ be a constant, D is called a $b-$cigar domain if each pair of points $x_1, x_2 \in D \setminus \{\infty\}$ can be joined by a curve $\gamma \subseteq D$ for which

$$\min_{j=1,2} dia(\gamma(x_j, x)) \leq bd(x, \partial D) \qquad \text{for all} \qquad x \in \gamma, \tag{1.2}$$

where $\gamma(x_j, x)$ is the part of γ between x_j and x, and $d(x, \partial D)$ is the Euclidean distance from x to ∂D, and $dia(\gamma)$ is the Euclidean diameter of γ.

We say that D is a cigar domain if D is a $b-$cigar domain for some $b > 0$.

F.W. Gehring proved the following result in [4].

Theorem A. *The following four conditions are equivalent each other:*

(1) D is a quasidisk;

(2) D is a linearly locally connected domain;

(3) There exists a constant $d > 1$ such that for each pair of points $x_1, x_2 \in \partial D \setminus \{\infty\}$,

$$\min_{j=1,2} dia(\gamma_j) \leq d|x_1 - x_2|, \tag{1.3}$$

where γ_1, γ_2 are the components of $\partial D \setminus \{x_1, x_2\}$.

(4) There exists a constant $b > 1$ such that each pair of points $x_1, x_2 \in D \setminus \{\infty\}$ can be joined by a rectifiable curve β in D for which

$$l(\beta) \leq b|x_1 - x_2|, \tag{1.4}$$

$$\min_{j=1,2} l(\beta(x_j, x)) \leq bd(x, \partial D) \tag{1.5}$$

for each $x \in \beta$. Here $l(\beta)$ denotes the Euclidean length of β.

2 Lemmas

In this section, we shall establish and introduce the following five lemmas which are crucial in the proof of our theorem 1.1 and theorem 1.2.

Lemma 2.1. *If D has the $c-$max property, then D^* is a $2a_0(2c+1)-$cigar domain, where a_0 is an absolute constant.*

Proof. For any finite points $z_1, z_2 \in D^*$, let γ be the hyperbolic geodesic which joins z_1 and z_2 in D^*, for any $z \in \gamma \setminus \{z_1, z_2\}$, suppose that $f\colon B^2 \to D^*$ is a conformal mapping with $f(0) = z$ and

$$f^{-1}(\gamma) \subseteq R = \{z : z \in R^2, \mathrm{Im} z = 0\},$$

where f^{-1} is the inverse of f. Then there exist

$$x_1 \in \{z : z \in S^1, \mathrm{Im} z > 0\} \quad \text{and} \quad x_2 \in \{z : z \in S^1, \mathrm{Im} z < 0\}$$

by [17,Corollary 10.3] such that $\alpha_j = f([0, x_j))$ is rectifiable with

$$l(\alpha_j) < a_0 d(z, \partial D^*), \quad j = 1, 2,$$

where a_0 is an absolute constant, and $[0, x_j)$ is the half open segment which joins the origin O and x_j, $j = 1, 2$.

Let $y_j = f(x_j)$, $\alpha = \alpha_1 \cup \alpha_2$, then $y_j \in \partial D$ and

$$l(\alpha) \leq l(\alpha_1) + l(\alpha_2) < 2a_0 d(z, \partial D^*).$$

For above $y_1, y_2 \in \partial D^* = \partial D$, there exist a simple curve $\beta \subseteq D$ which joins y_1 and y_2 such that

$$|y - y'| \leq c \max_{j=1,2} |y' - y_j| \quad \text{for all} \quad y \in \beta \quad \text{and} \quad y' \in \partial D \tag{2.1}$$

by D has the $c-$max property.

If taking $y' = y_1$ in (2.1), then we get

$$|y - y_1| \leq c|y_1 - y_2|$$

for any $y \in \beta$. This and the triangular inequality imply

$$dia(\beta) \leq 2c|y_1 - y_2| \leq 2cl(\alpha). \tag{2.2}$$

If we denote by D_0 the bounded domain with boundary $\alpha \cup \beta$, then one of the points z_1 and z_2 must be in D_0. Without loss of generality, we may assume that $z_1 \in D_0$, then we can obtain

$$\begin{aligned} dia(\gamma(z_1, z)) &\leq dia(D_0) = dia(\partial D_0) \\ &\leq dia(\alpha) + dia(\beta) \\ &\leq l(\alpha) + dia(\beta) \\ &\leq 2a_0(2c+1)d(z, \partial D^*). \end{aligned} \tag{2.3}$$

This yields

$$\min_{j=1,2} dia(\gamma(z_j, z)) \leq 2a_0(2c+1)d(z, \partial D^*),$$

hence D^* is a $2a_0(2c+1)-$cigar domain.

Lemma 2.2. *If D is a $c-$cigar domain, then $D \in (2c+2) - OLC$.*

Proof. Take $b = 2c+2$. If $D \notin b - OLC$, then there exist $y_0 \in R^2$, $0 < r < \infty$ and $x_1, x_2 \in D \setminus B^2(y_0, r)$, such that x_1 and x_2 can not be joined by any curve in $D \setminus B^2(y_0, r/b)$.

Since D is a $c-$cigar domain, there exists an arc $\gamma \subseteq D$ such that γ joining x_1 and x_2 with

$$\min_{j=1,2} dia(\gamma(x_j, x)) \leq cd(x, \partial D) \tag{2.4}$$

for all $x \in \gamma$.

It is obvious that $\gamma \cap S^1(y_0, r/b) \neq \emptyset$. If taking $y \in \gamma \cap S^1(y_0, r/b)$, then (2.4) implies

$$d(y, \partial D) \geq \frac{1}{c} \min_{j=1,2} dia(\gamma(x_j, y)) \geq \frac{1}{c} \min_{j=1,2} |x_j - y| \geq \frac{1}{c}(1 - \frac{1}{b})r. \tag{2.5}$$

But

$$d(y_0, \partial D) \leq \frac{r}{b}. \tag{2.6}$$

The above (2.5), (2.6) and the triangular inequality yield

$$\frac{1}{c}(1 - \frac{1}{b})r \leq d(y, \partial D) \leq |y - y_0| + d(y_0, \partial D) \leq \frac{2}{b}r,$$

$$b \leq 2c + 1, \tag{2.7}$$

(2.7) contradicts with $b = 2c + 2$. Hence $D \in (2c + 2) - OLC$.

Lemma 2.3. *If D^* is a c_0-cigar domain, then $D \in (16c_0 + 21) - ILC$.*

Proof. Take $\delta = 8c_0 + 10$. For any $u \in R^2$, $s > 0$, and $z_1, z_2 \in D \cap \overline{B}^2(u, s)$, $z_1 \neq z_2$. Denote $z = \frac{1}{2}(z_1 + z_2)$ and $r = |z_1 - z_2|$. Next we first prove that z_1, z_2 must be in the same component of $\overline{B}^2(z, \frac{1}{2}\delta r) \setminus D^*$.

If z_1, z_2 belong to different components of $\overline{B}^2(z, \frac{1}{2}\delta r) \setminus D^*$, then z_1, z_2 must be in the different components of $\overline{B}^2(z, \frac{1}{2}r) \setminus D^*$. Let β be the line segment which joins z_1 and z_2, then β contains a subcurve $\alpha \subseteq D^*$ such that α divide D^* into D_1 and D_2, and $dia(D_j) \geq \frac{1}{2}r(\delta - 1)$, $j = 1, 2$. This yields

$$\min_{j=1,2} dia(D_j) \geq \frac{1}{2}r(\delta - 1). \tag{2.8}$$

For any $x \in \alpha$, if $D_1 \nsubseteq B^2(x, (2c_0 + 2)dia(\alpha))$ and $D_2 \nsubseteq B^2(x, (2c_0 + 2)dia(\alpha))$, then take

$$x_j \in D_j \setminus \overline{B}^2(x, (2c_0 + 2)dia(\alpha)), \qquad j = 1, 2.$$

Since D^* is a c_0-cigar domain, there exists an arc $\gamma \subseteq D^*$ joining x_1 and x_2 with

$$\min_{j=1,2} dia(\gamma(x_j, w)) \leq c_0 d(w, \partial D^*) \tag{2.9}$$

for all $w \in \gamma$.

Take $y \in \gamma \cap S^1(x, dia(\alpha))$, then we can get

$$\begin{cases} |y - x_j| \geq (2c_0 + 1)dia(\alpha), \\ \min_{j=1,2} dia(\gamma(x_j, y)) \leq c_0 d(y, \partial D^*), \end{cases} \qquad j = 1, 2. \tag{2.10}$$

This implies

$$d(y, \partial D^*) \geq \frac{2c_0 + 1}{c_0} dia(\alpha). \tag{2.11}$$

But

$$d(x, \partial D^*) \leq dia(\alpha). \tag{2.12}$$

(2.11),(2.12) and the triangular inequality yield

$$\frac{2c_0+1}{c_0}dia(\alpha) \le d(y,\partial D^*) \le |y-x| + d(x,\partial D^*) \le 2dia(\alpha),$$

so

$$dia(\alpha) \le 0,$$

this is obviously impossible. Hence $D_1 \subseteq \overline{B}^2(x,(2c_0+2)dia(\alpha))$ or $D_2 \subseteq \overline{B}^2(x,(2c_0+2)dia(\alpha))$, and we can obtain

$$\min_{j=1,2} dia(D_j) \le 2(2c_0+2)dia(\alpha). \tag{2.13}$$

(2.8), (2.13) and $dia(\alpha) \le r$ imply

$$\delta \le 8c_0 + 9,$$

this contradicts with $\delta = 8c_0+10$. Hence z_1, z_2 must be in the same component of $\overline{B}^2(z, \frac{1}{2}\delta r) \setminus D^*$, and there exists an arc $\gamma \subseteq D$ joining z_1 and z_2 with

$$dia(\gamma) \le \delta r = \delta|z_1 - z_2| \le 2\delta s. \tag{2.14}$$

The above (2.14) implies

$$\begin{aligned} \gamma &\subseteq D \cap \overline{B}^2(u, s + dia(\gamma)) \subseteq D \cap \overline{B}^2(u,(2\delta+1)s) \\ &= D \cap \overline{B}^2(u,(16c_0+21)s). \end{aligned}$$

Hence $D \in (16c_0+21) - ILC$, this completes the proof of lemma 2.3.

Lemma 2.4. *Suppose that $f : \overline{R}^2 \to \overline{R}^2$ is a $K-$quasiconformal mapping with $f(\infty) = \infty$, for any $x \in R^2$, if $0 < r_1 \le r_2 < +\infty$, then*

$$\frac{L(x,f,r_2)}{l(x,f,r_1)} \le c\left(\frac{r_2}{r_1}\right)^K, \tag{2.15}$$

where $c = c(K)$ is a constant which depends only on K.

Proof For any $x \in R^2$ and $0 < r < +\infty$, since f is a $K-$quasiconformal mapping of $\overline{R}^2$, according to [18, p_{79}] we have

$$\frac{L(x,f,r)}{l(x,f,r)} \le c' \tag{2.16}$$

where $c' = c'(K)$ is a constant which depends only on K.

For any $0 < r_1 \le r_2 < +\infty$, from (2.16) we can get

$$\frac{L(x,f,r_2)}{l(x,f,r_1)} \le c'^2 \frac{l(x,f,r_2)}{L(x,f,r_1)}. \tag{2.17}$$

Without loss of generality, we may assume that $l(x,f,r_2) > L(x,f,r_1)$. Let Γ be the curve family which joins $S^1(x,r_1)$ and $S^1(x,r_2)$ in $B^2(x,r_2)\backslash\overline{B}^2(x,r_1)$, $M(\Gamma)$ is the module of Γ, Making use of the comparison principle of module and the properties of $K-$quasiconformal mappings in [18] we can obtain

$$\frac{2\pi}{\log\frac{l(x,f,r_2)}{L(x,f,r_1)}} \geq M(f(\Gamma)) \geq \frac{1}{K}M(\Gamma) = \frac{1}{K}\frac{2\pi}{\log\frac{r_2}{r_1}}. \tag{2.18}$$

Combining (2.17) and (2.18) we get

$$\frac{L(x,f,r_2)}{l(x,f,r_1)} \leq c'^2\left(\frac{r_2}{r_1}\right)^K, \tag{2.19}$$

from (2.19) we can obtain (2.15) with $c = c'^2$.

Lemma 2.5[8]**.** *Let $f:\overline{R}^2 \to \overline{R}^2$ be a homeomorphism with $f(\infty)=\infty$. If there exists a constant $c>0$ such that*

$$\left[dia\left(f\left(B^2(x,r)\right)\right)\right]^2 \leq c\cdot m\left[f\left(B^2(x,r)\right)\right] \tag{2.20}$$

for all $x\in R^2$ and $0<r<\infty$, then f is a quasiconformal mapping. Here $m\left[f\left(B^2(x,r)\right)\right]$ denotes the $2-$dimensional Lebesgue measure of $f\Big(B^2(x,r)\Big)$.

3 Proofs of theorems

Proof of theorem 1.1. The necessity. In fact the necessity was proved by F.W. Gehring and K. Hag in [5, theorem 2.7], for the sake of readability, we give the proof as follows.

If D is a quasidisk, then from the equaivalent condition 4 of theorem A and the corollary 2 in [6] we know that there exists a constant $b>1$ such that each pair of finite points $x_1,x_2\in D$ can be joined by the hyperbolic geodesic $\beta\subseteq D$ and β satisfies (1.4) and (1.5).

Next let Γ denote the family of hyperbolic geodesics in D. Then each pair of finite points $x_1,x_2\in D$ can be joined by a curve $\gamma\in\Gamma$. Fix $\gamma\in\Gamma$ and for any ordered triple of points $x_1,x,x_2\in\gamma$ and any $y\in\partial D\setminus\{\infty\}$, then $\beta=\gamma(x_1,x_2)\in\Gamma$ and by (1.4) and the triangle inequality we have

$$\begin{aligned}|x-y| &\leq \frac{1}{2}(|x-x_1|+|x_1-y|+|x-x_2|+|x_2-y|)\\ &\leq \frac{1}{2}(l(\beta)+|x_1-y|+|x_2-y|)\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2}(b|x_1 - x_2| + |x_1 - y| + |x_2 - y|) \\ &\leq \frac{b+1}{2}(|x_1 - y| + |x_2 - y|) \\ &\leq (b+1)\max_{j=1,2}|x_j - y|. \end{aligned}$$

Hence D has the $c-$max property with $c = b+1$ and Γ the family of hyperbolic geodesics in D.

Since D is a quasidisk, hence D^* is also a quasidisk. Makeing use of the same method as above we can get that D^* has the max property.

The sufficiency. Suppose that both D and D^* have the max property.

(1) Since D^* has the max property, hence D is a cigar domain by lemma 2.1, this and lemma 2.2 imply $D \in OLC$.

(2) since D has the max property, hence D^* is a cigar domain by lemma 2.1, this and lemma 2.3 imply $D \in ILC$.

From the above (1), (2) and the equivalent condition 2 of theorem A we know that D is a quasidisk.

Proof of theorem 1.2. The necessity. Suppose that D has the $c-$max property. Let $\Gamma \subseteq D$ be the family of curves which satisfies (1.1), $\Gamma' = f(\Gamma)$, then Γ' is a family of curves in D'. For any x_1', $x_2' \in D' \setminus \{\infty\}$, denote $x_1 = f^{-1}(x_1')$ and $x_2 = f^{-1}(x_2')$, then $x_1, x_2 \in D \setminus \{\infty\}$, and x_1, x_2 can be joined by a curve $\gamma \in \Gamma$ for which (1.1) is satisfied. Let $\gamma' = f(\gamma)$, then $\gamma' \in \Gamma'$ and γ' joins x_1' and x_2'.

Next for any ordered triple of points $x_1', x', x_2' \in \gamma'$ and each $y' \in \partial D' \setminus \{\infty\}$, let $x_1 = f^{-1}(x_1')$, $x = f^{-1}(x')$, $x_2 = f^{-1}(x_2')$ and $y = f^{-1}(y')$, then $x_1, x, x_2 \in \gamma$ are the ordered triple of points and $y \in \partial D \setminus \{\infty\}$ for which (1.1) is satisfied.

without loss of generality, we may assume that

$$|x - y| \leq c|x_1 - y|.$$

From this we can get

$$|f(x) - f(y)| \leq L(y, f, c|x_1 - y|). \tag{3.1}$$

On the other hand, it is obvious that

$$|f(x_1) - f(y)| \geq l(y, f, |x_1 - y|). \tag{3.2}$$

Making use of (3.1), (3.2) and lemma 2.4 we get

$$\frac{|f(x) - f(y)|}{|f(x_1) - f(y)|} \leq c^* c^K,$$

where c^* is a constant which depends only on K, and hence

$$|x' - y'| \le c^* c^K \max_{j=1,2} |x'_j - y'|,$$

this implies that D' has the $c^* c^K-$max property.

The sufficiency. For any $x \in R^2$ and $0 < r < \infty$, choose $y \in B^2(x,r)$ such that

$$dia\left(f(B^2(x,r))\right) \le 3|f(x) - f(y)|. \tag{3.3}$$

Theorem 1.1 implies that $(B^2(x,r))^*$ has the max property. By the assumption of theorem 1.2 we know that $f((B^2(x,r))^*)$ has the max property, hence $f(B^2(x,r))$ is a cigar domain by lemma 2.1. There exists a constant $b > 0$ and a curve $\gamma \subseteq D$ joins $f(x)$ and $f(y)$ such that

$$\min\{dia(\gamma(f(x),z)), dia(\gamma(f(y),z))\} \le bd(z, \partial(f(B^2(x,r)))) \tag{3.4}$$

for all $z \in \gamma$. If we choose $z_0 \in \gamma$ such that $dia(\gamma(f(x),z_0)) = dia(\gamma(f(y),z_0))$, then (3.3) and (3.4) imply

$$\begin{aligned} d(z_0, \partial(f(B^2(x,r)))) &\ge \frac{dia(\gamma(f(x),z_0))}{b} \\ &\ge \frac{dia(\gamma(f(x),f(y)))}{2b} \\ &\ge \frac{|f(x)-f(y)|}{2b} \\ &\ge \frac{dia(f(B^2(x,r)))}{6b}. \end{aligned}$$

This yields

$$B^2\left(z_0, \frac{dia(f(B^2(x,r)))}{6b}\right) \subseteq f(B^2(x,r)),$$

$$\pi\left(\frac{dia(f(B^2(x,r)))}{6b}\right)^2 \le m(f(B^2(x,r))),$$

and

$$\left(dia(f(B^2(x,r)))\right)^2 \le \frac{36b^2}{\pi} m(f(B^2(x,r))). \tag{3.5}$$

From (3.5) and lemma 2.5 we know that f is a quasiconformal mapping.

References

[1] K. Astala, M. Zinsmeister, Teichmüller space and BMOA, Math. Ann., 289(1991), 613-625.

[2] O.J. Broch, Quadrilaterals and John disks, comput. Methods Funct. Theory, 4(2004), 419-434.

[3] Z. Balogh, A. Volberg, Geometric localization, uniformly John property and separated semihyperbolic dynamics, Ark. Mat., 34(1996), 21-49.

[4] F.W. Gehring, Characteristic Properties of Quasidisks, Presses de l'Université de Montréal, Montréal, Que.(1982).

[5] F.W. Gehring, K. Hag, Remarks on uniform and quasiconformal extension domains, complex variables theory Appl., 9(1987), 175-188.

[6] F.W. Gehring, B.G. Osgood, Uniform domains and the quasi-hyperbolic metric, J. d'Analyse Math., 36(1979), 50-74.

[7] D.H. Hamilton, The closure of Teichmüller space, J. Analyse Math., 55(1990), 40-50.

[8] X.G. He, Uniform domains and quasiconformal mappings. Chinese Ann. Math., 13(1992), 66-69.

[9] K. Hag, P. Hag. John disks and the pre-Schwarzian derivative, Ann. Acad. Sci. Fenn. Math., 26(2001), 205-224.

[10] J. Heinonen, P. Koskela, Weighted Sobolev and Poincaré inequalities and quasiregular mappings of polynomial type, Math. Scand., 77(1995), 251-271.

[11] K. Kim, Relations between certain domains in the complex plane and polynomial approximation in the domains, Bull. Korean Math. Soc., 39(2002), 687-704.

[12] I. Kra, Families of univalent functions and Kleinian groups, Israel J. Math., 60(1987), 89-127.

[13] C.T. McMullen, Self-similarity of Siegel disks and Hausdorff dimension of Julia sets, Acta Math., 180(1998), 247-292.

[14] H. Miyachi, Cusps in complex boundaries of one-dimensional Teichmüller space, Cofrom. Geom. Dyn., 7(2003), 103-151.

[15] R. Näkki, J. Väisälä, John disks, Exposition. Math., 9(1991), 3-43.

[16] C.L. Petersen, S. Zakeri, On the Julia set of a typical quadratic polynomial with a Siegal disk, Ann. of Math. (2), 159(2004), 1-52.

[17] Ch. Pommerenke, Univalent Functions, Vandenhoeck, Götingen(1975).

[18] J. Väisälä, Lectures on $n-$Dimensional Quasiconformal Mappings, Springer-Verlag, New York(1971).

[19] J.A. Velling, Degeneration of quasicircles: inner and outer radii of Teichmüller sapces, Ann. Acad. Sci. Fenn. Ser. A I Math., 18(1993), 191-202.

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