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The Construction of a Quasiconformal Mapping[1] from Unit Ball onto Bounded Convex Domain in R^3

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Abstract

In this paper, the authors proved that a bounded convex domain D in R^3 is a quasiball by constructing a quasiconformal mapping from B^3 onto D, and hence the Riemann type mapping theorem of quasiconformal mapping holds on bounded convex domain in R^3.

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1. Introduction

An important problem in the theory of higher-dimensional quasiconformal mappings is to characterize in various ways those domains D in $\overline{R}^n$, the one point compactification of $n-$ space R^n, which can be mapped onto the unit ball B^n in R^n[5,6,10,12].

At present only two such criteria are known, both rather implicit. According to the first, a domain D in R^n is quasiconformally equivalent to B^n if and only if its boundary ∂D has a neighborhood U such that $D \cap U$ can be mapped

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quasiconformally into B^n so that ∂D corresponds to ∂B^n [3]. The second criterion states that a domain D in R^n is quasiconformally equivalent to B^n if and only if the corresponding $n-$dimensional Royden algebras $A_n(D)$ and $A_n(B^n)$ are isomorphic [1,2]. In particular, no purely geometric criterion, analogous to the Riemann mapping theorem, is known. Moreover, simple examples exist to show there can be no satisfactory characterization for domain D in R^n with the desired mapping property in terms of the tangential or smoothness properties of their boundaries[4,7].

For convenience we shall adopt the relatively standard notation for quasiconformal mapping given in [11]. Let D be a domain in $\bar{R}^n$, we call D is a quasiball if there exits a $K-$quasiconformal mapping of $\bar{R}^n$ such that $f(D) = B^n$.

In this paper, our main aim is to prove that a bounded convex domain D in R^3 is a quasiball by constructing quasiconformal mapping from B^3 onto D. Although F. W. Gehring and J. Väisälä had proved that any bounded convex domain D in R^3 is quasiconformally equivalent to B^3 in [9, section 5], but they could not obtain that D is a quasiball.

Theorem *If D is a convex domain in R^3, then D must be a quasiball in R^3.*

To prove the above theorem, we shall construct a homeomorphism f from the unit ball B^3 onto the bounded convex domain D in section 2, the original idea of the construction of homeomorphism f is due to F. W. Gehring and Ch. Pommerenke [8], in which they proved that a convex curve which separates the boundary circles of an annulus A is a quasicircle. In section 3, we shall estimate the dilatation $H(x, f)$ of f and prove that it is bounded uniformly in differentiable points of f. At last, we shall prove that f has the ACL property in section 4.

2. The construction of homeomorphism f

Since D is bounded in R^3, hence there exists $x_0 \in D$ such that $d(x_0, \partial D) = \max\limits_{x\in D} d(x, \partial D)$. If let $a = d(x_0, \partial D)$, $b = dia(D)$, then $B^3(x_0, \frac{1}{2}a) \subset D \subset B^3(x_0, b)$. Without loss of generality by similarity transformation, we may assume that $a = 2$, $b = c$, $x_0 = 0$, that is to say $B^3(0, 1) \subset D \subset B^3(0, c)$.

For convenience we shall adopt the spherical coordinates, for any $\theta_1 \in [0, \pi]$ and $\theta_2 \in [0, 2\pi]$, denotes $e^{i(\theta_1,\theta_2)}$ by the point $(\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_1)$ in R^3. for any $\theta_1 \in [0, \pi]$ and $\theta_2 \in [0, 2\pi]$, there exists unique point $x = r(\theta_1, \theta_2)e^{i(\theta_1,\theta_2)} \in \partial D$, $r(\theta_1, \theta_2) \in (1, c)$. Let $\theta_1' \in [0, \pi]$ and $\theta_2' \in [0, 2\pi]$ such that both $|\theta_1' - \theta_1|$ and $|\theta_2' - \theta_2|$ are very small, $x' = r(\theta_1', \theta_2')e^{i(\theta_1',\theta_2')} \in \partial D$, $r(\theta_1', \theta_2') \in (0, c)$.

Denotes T by the two dimensional plane which through the origin O, points x and x'. Let $B^2(0, c) = T \cap B^3(0, c)$ and $B^2(0, 1) = T \cap B^3(0, 1)$, L_1 and L_2

are the two tangent rays drawn from $\partial B^2(0,1)$ through x to ∞ in T, A and B are the two tangent points, where the amplitude of point A is larger than the amplitude of point x in T and the amplitude of point B is less than the amplitude of point x in T. Denotes L by the straight line which through the origin O and the point x' in T, P and Q are the intersection points of L and L_1, L and L_2 respectively.

Since D is convex domain and $|x'-x|$ is small enough, hence x' must be in the line segment which with end points P and Q. If let $\theta = \angle x'ox$, then θ can be decided by the following

$$\sin\frac{\theta}{2} = \frac{1}{2}|e^{i(\theta_1',\theta_2')} - e^{i(\theta_1,\theta_2)}|. \tag{1}$$

But

$$|r(\theta_1',\theta_2') - r(\theta_1,\theta_2)| \le |x'-x| \le \max\{|x-P|,|x-Q|\} \tag{2}$$

and

$$\max\{|x-P|,|x-Q|\} = \frac{r^2(\theta_1,\theta_2)\tan\theta}{1-\sqrt{r^2(\theta_1,\theta_2)-1}\tan\theta}. \tag{3}$$

Combining (2) and (3), we have

$$|r(\theta_1',\theta_2') - r(\theta_1,\theta_2)| \le \frac{r^2(\theta_1,\theta_2)\tan\theta}{1-\sqrt{r^2(\theta_1,\theta_2)-1}\tan\theta}. \tag{4}$$

It is easy to obtain

$$\limsup_{\substack{\theta_1'\to\theta_1\\ \theta_2'\to\theta_2}} \frac{|r(\theta_1',\theta_2') - r(\theta_1,\theta_2)|}{|e^{i(\theta_1',\theta_2')} - e^{i(\theta_1,\theta_2)}|} \le r^2(\theta_1,\theta_2) < cr(\theta_1,\theta_2) < c^2 \tag{5}$$

by (1) and (4).

When both $|\theta_1'-\theta_1|$ and $|\theta_2'-\theta_2|$ are small enough, we can get

$$|e^{i(\theta_1',\theta_2')} - e^{i(\theta_1,\theta_2)}| \le 4\left(\left|\sin\frac{\theta_1'-\theta_1}{2}\right| + \sin\theta_1\left|\sin\frac{\theta_2'-\theta_2}{2}\right|\right) \tag{6}$$

and

$$\limsup_{\substack{\theta_1'\to\theta_1\\ \theta_2'\to\theta_2}} \frac{|\sin\frac{\theta_1'-\theta_1}{2}| + \sin\theta_1|\sin\frac{\theta_2'-\theta_2}{2}|}{|\frac{\theta_1'-\theta_1}{2}| + \sin\theta_1|\frac{\theta_2'-\theta_2}{2}|} \le 2. \tag{7}$$

Combining (5), (6) and (7), we get

$$\limsup_{\substack{\theta_1'\to\theta_1\\ \theta_2'\to\theta_2}} \frac{|r(\theta_1',\theta_2') - r(\theta_1,\theta_2)|}{|\theta_1'-\theta_1| + \sin\theta_1|\theta_2'-\theta_2|} \le 4r^2(\theta_1,\theta_2). \tag{8}$$

Constructing homeomorphism f as following

$$\begin{cases} f(se^{i(\theta_1,\theta_2)}) = sr(\theta_1,\theta_2)e^{i(\theta_1,\theta_2)}, \quad 0 \leq s < \infty, \theta_1 \in [0,\pi], \theta_2 \in [0,2\pi], \\ f(\infty) = \infty. \end{cases} \tag{9}$$

It is easy to see that f is a homeomorphism of $\bar{R}^3$ onto $\bar{R}^3$ and $f(B^3) = D$, $f(\partial B^3) = \partial D$. Next we shall prove that f is differentiable a.e. in R^3. For any $x = se^{i(\theta_1,\theta_2)}$ and $x' = s'e^{i(\theta'_1,\theta'_2)}$, we have

$$|f(x') - f(x)| = |r(\theta'_1,\theta'_2)x' - r(\theta_1,\theta_2)x| \leq s'|r(\theta'_1,\theta'_2) - r(\theta_1,\theta_2)| + r(\theta_1,\theta_2)|x' - x|. \tag{10}$$

Using (5) and (10) we can get

$$\begin{aligned} \limsup_{x'\to x} \frac{|f(x') - f(x)|}{|x' - x|} &\leq r(\theta_1,\theta_2) + \limsup_{\substack{\theta'_1\to\theta_1 \\ \theta'_2\to\theta_2 \\ s'\to s}} \frac{|r(\theta'_1,\theta'_2) - r(\theta_1,\theta_2)|}{|\frac{s}{s'}e^{i(\theta_1,\theta_2)} - e^{i(\theta'_1,\theta'_2)}|} \\ &\leq r(\theta_1,\theta_2) + 2\limsup_{\substack{\theta'_1\to\theta_1 \\ \theta'_2\to\theta_2}} \frac{|r(\theta'_1,\theta'_2) - r(\theta_1,\theta_2)|}{|e^{i((\theta_1,\theta_2)} - e^{i(\theta'_1,\theta'_2)}|} \\ &\leq r(\theta_1,\theta_2) + 2r^2(\theta_1,\theta_2) < c + 2c^2. \end{aligned} \tag{11}$$

We know that f is differentiable a.e. in R^3 by (11) and the Rademacher-Stepanov theorem [11].

3. The estimation of dilatation $H(x, f)$

For any $x \in R^3$, the dilatation $H(x, f)$ of f at x is defined by

$$H(x,f) = \limsup_{r\to 0} \frac{\max\limits_{|y-x|=r} |f(y) - f(x)|}{\min\limits_{|y-x|=r} |f(y) - f(x)|}. \tag{12}$$

Next we assume that f is differentiable at x. Let

$$x = (x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}e^{i(\theta_1,\theta_2)},$$

$$f(x) = r(\theta_1,\theta_2)x = (u(x_1,x_2,x_3), v(x_1,x_2,x_3), w(x_1,x_2,x_3)),$$

$$u_i = \frac{\partial u(x_1,x_2,x_3)}{\partial x_i}, v_i = \frac{\partial v(x_1,x_2,x_3)}{\partial x_i}, w_i = \frac{\partial w(x_1,x_2,x_3)}{\partial x_i}, i = 1,2,3.$$

Denote

$$E = u_1^2 + v_1^2 + w_1^2, F = u_2^2 + v_2^2 + w_2^2, G = u_3^2 + v_3^2 + w_3^2,$$

$$A = u_1u_2 + v_1v_2 + w_1w_2, B = u_1u_3 + v_1v_3 + w_1w_3, C = u_2u_3 + v_2v_3 + w_2w_3,$$
$$r = r(\theta_1, \theta_2), r_i = \frac{\partial r(\theta_1, \theta_2)}{\partial \theta_i}, i = 1, 2.$$

By calculating carefully it is not difficult to obtain

$$\left\{\begin{aligned}
&u_1 = r + r_1 \sin\theta_1 \cos\theta_1 \cos^2\theta_2 - r_2 \sin\theta_2 \cos\theta_2,\\
&v_1 = r_1 \sin\theta_1 \sin\theta_2 \cos\theta_1 \cos\theta_2 - r_2 \sin^2\theta_2,\\
&w_1 = r_1 \cos^2\theta_1 \cos\theta_2 - r_2 \cos\theta_1 \sin\theta_2/\sin\theta_1.
\end{aligned}\right.$$
$$\left\{\begin{aligned}
&u_2 = r_1 \sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 + r_2 \cos^2\theta_2,\\
&v_2 = r + r_1 \sin\theta_1 \sin^2\theta_2 \cos\theta_1 + r_2 \sin\theta_2 \cos\theta_2,\\
&w_2 = r_1 \cos^2\theta_1 \sin\theta_2 + r_2 \cos\theta_1 \cos\theta_2/\sin\theta_1.
\end{aligned}\right.$$
$$\left\{\begin{aligned}
&u_3 = -r_1 \sin^2\theta_1 \cos\theta_2,\\
&v_3 = -r_1 \sin^2\theta_1 \sin\theta_2,\\
&w_3 = r - r_1 \sin\theta_1 \cos\theta_1.
\end{aligned}\right.$$
$$\left\{\begin{aligned}
&E = r^2 + 2rr_1 \sin\theta_1 \cos\theta_1 \cos^2\theta_2 - 2rr_2 \sin\theta_2 \cos\theta_2 + r_1^2 \cos^2\theta_1 \cos^2\theta_2\\
&\qquad -2r_1r_2 \cos\theta_1 \sin\theta_2 \cos\theta_2/\sin\theta_1 + r_2^2 \sin^2\theta_2/\sin^2\theta_1,\\
&F = r^2 + 2rr_1 \sin\theta_1 \sin^2\theta_2 \cos\theta_1 + 2rr_2 \sin\theta_2 \cos\theta_2 + r_1^2 \cos^2\theta_1 \sin^2\theta_2\\
&\qquad +2r_1r_2 \cos\theta_1 \sin\theta_2 \cos\theta_2/\sin\theta_1 + r_2^2 \cos^2\theta_2/\sin^2\theta_1,\\
&G = r^2 - 2rr_1 \sin\theta_1 \cos\theta_1 + r_1^2 \sin^2\theta_1.
\end{aligned}\right.$$

$$\left\{\begin{aligned}
&A = 2rr_1 \sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 + rr_2(\cos^2\theta_2 - \sin^2\theta_2) + r_1^2 \cos^2\theta_1 \sin\theta_2 \cos\theta_2\\
&\qquad +r_1r_2 \cos\theta_1(\cos^2\theta_2 - \sin^2\theta_2)/\sin\theta_1 - r_2^2 \sin\theta_2 \cos\theta_2/\sin^2\theta_1,\\
&B = rr_1 \cos\theta_2(\cos^2\theta_1 - \sin^2\theta_1) - rr_2 \cos\theta_1 \sin\theta_2/\sin\theta_1\\
&\qquad -r^2 \sin\theta_1 \cos\theta_1 \cos\theta_2 + r_1r_2 \sin\theta_2,\\
&C = rr_1 \sin\theta_2(\cos^2\theta_1 - \sin^2\theta_1) + rr_2 \cos\theta_1 \cos\theta_2/\sin\theta_1\\
&\qquad -r_1^2 \sin\theta_1 \cos\theta_1 \sin\theta_2 - r_1r_2 \cos\theta_2.
\end{aligned}\right.$$

Considering the following characteristic equation

$$\begin{vmatrix} \lambda - E & -A & -B \\ -A & \lambda - F & -C \\ -B & -C & \lambda - G \end{vmatrix} = 0$$

or

$$\begin{aligned}
\lambda^3 - (E + F + G)\lambda^2 + (EF + EG + FG - A^2 - B^2 - C^2)\lambda&\\
-(2ABC + EFG - GA^2 - FB^2 - EC^2) = 0.& \qquad (13)
\end{aligned}$$

Suppose that λ_1, λ_2 and λ_3 are the three roots of the equation (13), without loss of generality, we may assume that $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$, we have

$$\left\{\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 = E + F + G,\\
&\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = EF + EG + FG - A^2 - B^2 - C^2,\\
&\lambda_1\lambda_2\lambda_3 = EFG + 2ABC - GA^2 - FB^2 - EC^2.
\end{aligned}\right. \qquad (14)$$

Using (12) and (14) we get

$$\begin{aligned} H(x,f) &= \frac{\max\{\lambda_1,\lambda_2,\lambda_3\}}{\min\{\lambda_1,\lambda_2,\lambda_3\}} = \frac{\lambda_3}{\lambda_1} \\ &< \frac{\lambda_3}{\lambda_1}+\frac{\lambda_2}{\lambda_1}+\frac{\lambda_3}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3} \\ &= \frac{(E+F+G)(EF+EG+FG-A^2-B^2-C^2)}{EFG+2ABC-(GA^2+FB^2+EC^2)}. \end{aligned} \tag{15}$$

Using the expression of E,F,G,A,B and C, direct computation yields

$$\begin{cases} (E+F+G)(EF+EG+FG-A^2-B^2-C^2) \\ \qquad = (3r^2+r_1^2+\frac{r_2^2}{\sin^2\theta_1})(3r^4+r^2r_1^2+\frac{r^2r_2^2}{\sin^2\theta_1}), \\ EFG+2ABC-(GA^2+FB^2+EC^2)=r^6. \end{cases} \tag{16}$$

By (8) we have

$$\begin{cases} r_1 \le 4r^2 < 4cr < 4c^2, \\ r_2 \le 4\sin\theta_1 r^2 < 4c\sin\theta_1 r < 4c^2\sin\theta_1. \end{cases} \tag{17}$$

Combining (15), (16) and (17) yields

$$H(x,f) < (3+32c^2)^2.$$

4. The ACL property of f

For any line segment l with end points A and B in R^3, parallel to the x_2-axis. Let O be the coordinate center, taking $b=\max\{|OA|,|OB|\}$, for any $z=se^{i(\theta_1,\theta_2)}$, $z'=s'e^{i(\theta_1',\theta_2')}\in l$ such that $|z-z'|<L/(2\sqrt{c^2-1})$, where L is the vertical distance from the coordinate center O to the line segment l. According to the construction of f in (9) and the triangle inequality, we have

$$\begin{aligned} |f(z')-f(z)| &\le r(\theta_1,\theta_2)|z'-z|+|z'||r(\theta_1',\theta_2')-r(\theta_1,\theta_2)| \\ &\le c|z'-z|+b|r(\theta_1',\theta_2')-r(\theta_1,\theta_2)|. \end{aligned} \tag{18}$$

On the other hand, using (1) and (4), it is easy to verify

$$\tan\theta \le \frac{|z'-z|}{L}. \tag{19}$$

Combining (4), (18), (19) and $|z'-z|\le L/(2\sqrt{c^2-1})$ yields

$$\begin{aligned} |f(z')-f(z)| &\le c|z'-z|+2br^2(\theta_1,\theta_2)|z'-z|/L \\ &\le (c+2bc^2/L)|z'-z|. \end{aligned} \tag{20}$$

For any $\varepsilon > 0$, we need only to take $\delta = \min\{L/(2\sqrt{c^2-1}), \varepsilon/(c+2bc^2/L)\}$, for any $z_i, z_i' \in l$, if $\sum\limits_i |z_i - z_i'| < \delta$, then $\sum\limits_i |f(z_i) - f(z_i')| < \varepsilon$. Hence f is absolutely continuous on the line segment l.

Similarly, using the same way we can prove that f is absolutely continuous on the line segment which parallel to the x_1-axis or x_3-axis in R^3.

According to the above analysis and the analytic definition for quasiconformal mapping in [11], we know that f is a quasiconformal mapping of $\bar{R}^3$ onto $\bar{R}^3$, and hence the bounded convex domain D is a quasiball in R^3.

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