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Viscosity Approximation Fixed Points for Nonexpansive Nonself-Mapping in Banach Space[1]

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Abstract. Let X be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping from X to X^*, and C be a closed convex subset of X which is also a sunny nonexpansive retract of X, and $T : C \to X$ a non-expansive mapping satisfying the weakly inward condition and $F(T) \neq \emptyset$, and $f : C \to C$ be a fixed contractive mapping. The sequence $\{x_n\}$ is given by

$$x_{n+1} = P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n)),$$

where $\alpha_n,\ \beta_n \in (0,1)$, and P is sunny nonexpansive retract of X onto C. We prove that $\{x_n\}$ strongly converges to a fixed point of T as α_n and β_n satisfying some appropriate conditions.

Keywords: Fixed point; nonexpansive mapping; reflexive Banach space; weakly sequentially continuous duality mapping

1. Introduction

Let X be a real Banach space and let J denote the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\,\|f\|\,,\ \|x\| = \|f\|\},\ \forall x \in X,$$

where X^* denotes the dual space of X and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued duality mapping by j. And let C be a nonempty closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping if for all $x, y \in C$, such that

$$\|Tx - Ty\| \leq \|x - y\|\,,$$

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We use $F(T)$ to denote the set of fixed points of T; i.e., $F(T) = \{x \in C : x = Tx\}$. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\beta \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \beta \|x - y\|, \ x, y \in C.$$

In 2004, Hong-Kun Xu[1] defined the following one viscosity iteration for non-expansive mappings in uniformly smooth Banach space:

Theorem 1.1. *[1, Theorem 4.1, page 287]Let X be a uniformly smooth Banach space, C be a closed convex subset of X, $T : C \to C$ a non-expansive mapping with $F(T) \neq \phi$, and $f \in \Pi_C$, where Π_C denotes the set of all contractions on C. Then $\{x_t\}$ defined by the following:*

$$x_t = tf(x_t) + (1 - t)Tx_t, \ x \in C$$

converges strongly to a point in Fix(T). If we define $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t \to 0} x_t, \ f \in \Pi_C,$$

then Q(f) solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \ f \in \Pi_C, \ p \in F(T).$$

In [2], Schu introduced the iterative process below and proved the following theorem:

Theorem 1.2. *[2, Theorem 2.4, page 113] Let C be a nonempty closed convex and bounded subset of a Hilbert space H; Let $T : C \to C$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$; $\{\lambda_n\}_{n \in N} \subset (0, 1)$ with $\lim_{n\to\infty} \lambda_n = 1$; $\{\alpha_n\}_{n \in N} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ such that $(\{\alpha_n\}, \{\mu_n\})$ has property (A), $\{(1-\mu_n)(1-\lambda_n)^{-1}\}$ is bounded, and $\lim_{n\to\infty} \frac{1-\mu_n}{\alpha_n} = 0$, where $k_n := (1 + \alpha_n^2(1 + L)^2)^{\frac{1}{2}}$ and $\mu_n := \frac{\lambda_n}{k_n}$, $\forall n \in N$; fix an arbitrary point $w \in C$, and define that for all $n \in N$,*

$$x_{n+1} := \mu_{n+1}(\alpha_n T x_n + (1 - \alpha_n)x_n) + (1 - \mu_{n+1})w.$$

Then $\{x_n\}_n$ converges strongly to the unique fixed point of T closed to w.

Here the pair of sequences $(\{\alpha_n\}_n, \{\mu_n\}_n) \subset (0, \infty) \times (0, 1)$ is said to have property (A) if and only if the following conditions hold:
(1)$\{\alpha_n\}_n$ is decreasing;
(2)$\{\mu_n\}_n$ is strictly increasing;
(3)There exists a strictly increasing sequence $\{\beta_n\}_n \subset \mathbb{N}$ such that

(a)$\lim_n \frac{\alpha_n - \alpha_{n+\beta_n}}{1-\mu_n} = 0$;

(b)$\lim_n (1 - \mu_{n+\beta_n})(1 - \mu_n)^{-1} = 1$;

(c)$\lim_n \beta_n(1 - \mu_n) = \infty$.

In [9], recently Yi-sheng Song and Ru-dong Chen extended Theorem 1.1 to nonexpansive nonself-mapping in a reflexive Banach space: for $t \in (0,1)$,

$$x_t = P(tf(x_t) + (1-t)Tx_t) \tag{1.1}$$

and proved that $\{x_t\}$ strongly converges to a fixed point of T as $t \to 0$.

Let C be a closed convex subset of a reflexive Banach space X with a weakly sequentially continuous duality, the purpose of this paper is to use the following iteration process:$x_0 \in C$,

$$x_{n+1} = P[\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n)] \tag{1.2}$$

where $\{\alpha_n\}_n$, $\{\beta_n\}_n$ are sequences in (0,1) and $f : C \to C$ is a fixed contractive mapping, to approximate to the fixed point of non-expansive mapping T, which extends and improves several recent results.

2. Preliminaries

Recall that a Banach space is said to be smooth if the duality mapping J is singed valued. If C and D are nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is called a retraction from C to D if $P^2 = P$. It is easily known that a mapping $P : C \to D$ is retraction, then $Px = x$, $\forall x \in D$. A mapping $P : C \to D$ is called sunny if

$$P(Px + t(x - Px)) = Px, \ \forall x \in C$$

whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D. For more detail, see[3,4,5].

The following Lemma is well known (referece [1], [8]):

Lemma 2.1. *Let C be a nonempty convex subset of a smooth Banach space X, $D \subset C$, $J : X \to X^*$ be the (normalized) duality mapping of X, and $P : C \to D$ a retraction. Then the following are equivalent:*

$(i)\langle x - Px, j(y - Px)\rangle \leq 0$

$(ii)P$ is both sunny and nonexpansive.

Let C be a nonempty convex subset of a Banach space X, then for $x \in C$, we define the inward set[5]:

$$I_C(x) = \{y \in X : y = x + \lambda(z - x), \ z \in C \ and \ \lambda \geq 0\}.$$

A mapping $T : C \to E$ is said to be satisfying the inward condition if $Tx \in I_C(x)$ for all $x \in C$. T is also said to be satisfying the weakly inward condition if for each $x \in C$, $Tx \in \overline{I_C(x)}$ ($\overline{I_C(x)}$ is the closure of $I_C(x)$). Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C does.

Using Lemma2.1 and above these definitions, we can easily show the following Lemma (more details see reference [9]):

Lemma 2.2. *Let C be a nonempty closed subset of a smooth Banach space E, and $T : C \to E$ be nonexpansive nonself-mapping satisfying the weakly inward condition, and P be a sunny nonexpansive retraction of E onto C. Then $F(T) = F(PT)$.*

Lemma 2.3. *[9, Theorem 2.2]Let X be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a closed convex subset of X which is also a sunny nonexpansive retract of X, and $T : C \to X$ a non-expansive mapping satisfying the weakly inward condition and $F(T) \neq \emptyset$, and $f : C \to C$ be a fixed contractive mapping. $t \in (0,1)$, let $\{x_t\}$ be defined by (1.1), where P is a sunny nonexpansive retract of X onto C. Then as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point q of T such that q is the unique solution in $F(T)$ to the following variational inequality:*

$$\langle (I-f)q, j(q-u)\rangle \leq 0,\ \forall u \in F(T).$$

If Banach space X admits sequentially continuous duality mapping J from weak topology to weak star topology, by Lemma 1 of reference [10], we get that duality mapping J is single-valued. In the case, duality mapping J is also said to be weakly sequentially continuous, i.e. for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, then $J(x_n) \rightharpoonup^* J(x)$ (reference [8, 10]).

A Banach space X is said to be satisfying Opial's condition if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ $(n \to \infty)$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,\ \forall y \in X \textit{ with } x \neq y.$$

By Theorem 1 of reference [10], we know that if X admits a weakly sequentially continuous duality mapping, X satisfies Opial's condition.

Lemma 2.4. *Let C be a nonempty closed convex subset of a reflexive Banach space X which satisfying Opial's condition, and suppose $T : C \to X$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, i.e.*

$$x_n \rightharpoonup x,\ x_n - Tx_n \to 0 \textit{ implies } x = Tx.$$

The following lemmas will be needed in the sequel. Lemma2.5 is well know, (see, e.g., [1], [6]). The proof of Lemma2.5 can be deduced from [7, Lemma 2.5]:

Lemma 2.5. *Let X be an arbitrary real Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \tag{1.2}$$

for all $x, y \in X$ and for all $j(x+y) \in J(x+y)$.

Lemma 2.6. *Let $\{a_n\}_n$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n,\ n \in N, \tag{1.3}$$

where $\{\alpha_n\}_n \subset [0,1]$, and $\sum_{n=0}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \beta_n = 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property,*

$$a_{n+1} \le (1-\gamma_n)a_n + b_n, \ n \ge 0,$$

where $\{\gamma_n\} \subset (0,1)$, and $\{b_n\}$ are such that:

(i) $\lim\limits_{n\to\infty} \gamma_n = 0$, *and* $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\sum\limits_{n=0}^{\infty} |b_n| < +\infty$.

3. Main Results

Theorem 3.1. *Let X be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping J from X to X^*. Suppose C is a closed convex subset of X which is also a sunny nonexpansive retract of X, and $T : C \to X$ a non-expansive mapping satisfying the weakly inward condition and $F(T) \ne \emptyset$, and $f : C \to C$ be a fixed contractive mapping. The sequence $\{x_n\}$ is defined by (1.2), where P is sunny nonexpansive retract of X onto C, and $\alpha_n, \ \beta_n \in (0,1)$, and satisfy the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim\limits_{n\to\infty} \beta_n = 0$;

(iv) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < +\infty$;

(v) $\sum_{n=0}^{\infty} |\beta_n - \beta_{n-1}| < +\infty$.

Then as $n \to \infty$, the sequence $\{x_n\}$ converges strongly to a fixed point q of T such that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)q, j(q-u)\rangle \le 0, \ \forall u \in F(T).$$

Proof. First we show $\{x_n\}$ is bounded. Take $u \in F(T)$, it follows that

$$\begin{aligned}
\|x_{n+1} - u\| =& \|P(\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n)) - Pu\| \\
\le& \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n) - u\| \\
\le& \alpha_n \|f(x_n) - u\| + (1-\alpha_n)\|\beta_n(x_n - u) + (1-\beta_n)(Tx_n - u)\| \\
\le& \alpha_n\beta \|x_n - u\| + \alpha_n \|f(u) - u\| + (1-\alpha_n)\beta_n \|x_n - u\| \\
& + (1-\beta_n)(1-\alpha_n)\|x_n - u\| \\
=& (1-(1-\beta)\alpha_n)\|x_n - u\| + \alpha_n \|f(u) - u\| \\
\le& \max\{\|x_n - u\|, \frac{1}{1-\beta}\|f(u) - u\|\},
\end{aligned}$$

By induction,

$$\|x_n - u\| \le \max\{\|x_0 - u\|, \frac{1}{1-\beta}\|f(u) - u\|\}, \ n \ge 0.$$

Therefore $\{x_n\}$ is bounded, so are $\{Tx_n\}$ and $\{f(x_n)\}$.

Then we have

$$\begin{aligned}\|x_{n+1}-PTx_n\| &= \|P(\alpha_n f(x_n)+(1-\alpha_n)(\beta_n x_n+(1-\beta_n)Tx_n))-PTx_n\|\\ &\le \|\alpha_n f(x_n)+(1-\alpha_n)(\beta_n x_n+(1-\beta_n)Tx_n)-Tx_n\|\\ &\le \alpha_n\|f(x_n)-Tx_n\|+(1-\alpha_n)\|\beta_n x_n+(1-\beta_n)Tx_n-Tx_n\|\\ &= \alpha_n\|f(x_n)-Tx_n\|+(1-\alpha_n)\beta_n\|x_n-Tx_n\|\\ &\quad \to 0,\ (n\to\infty).\end{aligned}$$

We claim that

$$\|x_{n+1}-x_n\|\to 0,\ as\ n\to\infty.$$

Indeed we have

$$\begin{aligned} & \|x_{n+1}-x_n\|\\ = & \|P(\alpha_n f(x_n)+(1-\alpha_n)(\beta_n x_n+(1-\beta_n)Tx_n))\\ & -P(\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})(\beta_{n-1}x_{n-1}+(1-\beta_{n-1})Tx_{n-1}))\|\\ \le & \alpha_n\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\,\|f(x_{n-1})\|\\ & +(1-\alpha_n)\|\beta_n x_n+(1-\beta_n)Tx_n-\beta_{n-1}x_{n-1}-(1-\beta_{n-1})Tx_{n-1}\|\\ & +|\alpha_n-\alpha_{n-1}|\,\|\beta_{n-1}x_{n-1}+(1-\beta_{n-1})Tx_{n-1}\|\\ \le & \alpha_n\beta\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\,\|f(x_{n-1})\|+(1-\alpha_n)[\beta_n\|x_n-x_{n-1}\|\\ & +|\beta_n-\beta_{n-1}|\,\|x_{n-1}\|+(1-\beta_n)\|x_n-x_{n-1}\|\\ & +|\beta_n-\beta_{n-1}|\,\|Tx_{n-1}\|]+|\alpha_n-\alpha_{n-1}|\,\|\beta_{n-1}x_{n-1}+(1-\beta_{n-1})Tx_{n-1}\|\\ = & \alpha_n\beta\|x_n-x_{n-1}\|+(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|[\|f(x_{n-1})\|\\ & +\|\beta_{n-1}x_{n-1}+(1-\beta_{n-1})Tx_{n-1}\|]+(1-\alpha_n)|\beta_n-\beta_{n-1}|[\|x_{n-1}\|+\|Tx_{n-1}\|]\\ = & (1-(1-\beta)\alpha_n)\|x_n-x_{n-1}\|+M_n.\end{aligned}$$

where $M_n=|\beta_n-\beta_{n-1}|(1-\alpha_n)[\|x_{n-1}\|+\|Tx_{n-1}\|]+|\alpha_n-\alpha_{n-1}|(\|f(x_{n-1})\|+\|\beta_{n-1}x_{n-1}+(1-\beta_{n-1})Tx_{n-1}\|)$

From the bounded-ness of $\{x_n\}$, there exists a constant $M_1>0$ such that

$$M_n<M_1(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|),$$

thus,

$$\|x_{n+1}-x_n\|\le(1-(1-\beta)\alpha_n)\|x_n-x_{n-1}\|+M_1(|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|).$$

From conditions (i) (ii),(iv) and (v), we obtain from Lemma 2.7 that

$$\|x_{n+1}-x_n\|\to 0\ (n\to\infty).$$

Therefore we get

$$\begin{aligned}\|x_n-PTx_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-PTx_n\|\\ &\quad\to 0\ (n\to\infty).\end{aligned}\tag{3.1}$$

Let $q=\lim_{t\to 0}x_t$, where $\{x_t\}$ is defined by (1.1), from Lemma 2.3, we get that q is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I-f)q, j(q-u)\rangle\le 0,\ \forall u\in F(T)\tag{3.2}$$

Next we show

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle\le 0.\tag{3.3}$$

Indeed we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle = \limsup_{k\to\infty}\langle f(q)-q, j(x_{n_k}-q)\rangle$$

we may assume that $x_{n_k} \rightharpoonup x^*$ by X reflexive and $\{x_n\}$ bounded. It follows from Lemma 2.4, Lemma 2.2 and (3.1) that $x^* \in F(T) = F(PT)$. Hence by (3.2) and the duality mapping J is weakly sequentially continuous from X to X^*, we obtain,

$$\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle = \langle f(q)-q, j(x^*-q)\rangle \leq 0$$

Finally we show that $x_n \to q$.

As a matter of fact,

$$x_{n+1} - (\alpha_n f(x_n) + (1-\alpha_n)q) = (x_{n+1}-q) - \alpha_n(f(x_n)-q)$$

By Lemma 2.5 we have,

$$\begin{aligned}
&\|x_{n+1}-q\|^2\\
&= \|x_{n+1} - (\alpha_n f(x_n) + (1-\alpha_n)q) + \alpha_n(f(x_n)-q)\|^2\\
&\leq \|x_{n+1} - P(\alpha_n f(x_n) + (1-\alpha_n)q)\|^2 + 2\alpha_n\langle f(x_n)-q, j(x_{n+1}-q)\rangle\\
&\leq \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n x_n + (1-\beta_n)Tx_n) - (\alpha_n f(x_n) + (1-\alpha_n)q)\|^2\\
&+ 2\alpha_n\langle f(x_n)-f(q), j(x_{n+1}-q)\rangle + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\\
&\leq (1-\alpha_n)^2[\beta_n\|x_n-q\| + (1-\beta_n)\|Tx_n-q\|]^2\\
&+ 2\alpha_n\langle f(x_n)-f(q), j(x_{n+1}-q)\rangle + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\\
&\leq (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n\|f(x_n)-f(q)\|\|x_{n+1}-q\|\\
&+ 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle\\
&\leq (1-\alpha_n)^2\|x_n-q\|^2 + \alpha_n(\|f(x_n)-f(q)\|^2 + \|x_{n+1}-q\|^2)\\
&+ 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&(1-\alpha_n)\|x_{n+1}-q\|^2\\
\leq&(1-2\alpha_n+\alpha_n^2)\|x_n-q\|^2 + \alpha_n\beta^2\|x_n-q\|^2 + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q)\rangle.
\end{aligned}$$

i.e.

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq (1-\frac{1-\beta^2}{1-\alpha_n}\alpha_n)\|x_n-q\|^2 + \frac{\alpha_n^2}{1-\alpha_n}\|x_n-q\|^2\\
&+ \frac{2\alpha_n}{1-\alpha_n}\langle f(q)-q, j(x_{n+1}-q)\rangle\\
&\leq (1-\gamma_n)\|x_n-q\|^2 + \lambda\gamma_n\alpha_n + \frac{2}{1-\beta^2}\gamma_n\langle f(q)-q, j(x_{n+1}-q)\rangle.
\end{aligned}$$

Where $\gamma_n = \frac{1-\beta^2}{1-\alpha_n}\alpha_n$ and λ a constant such that $\lambda > \frac{1}{1-\beta^2}\|x_n-q\|^2$.

Hence,

$$\|x_{n+1} - q\|^2 \le (1 - \gamma_n)\|x_n - q\|^2 + \gamma_n(\lambda\alpha_n + \frac{2}{1-\beta^2}\langle f(q) - q, j(x_{n+1} - q)\rangle). \tag{3.4}$$

It is easily seen that $\gamma_n \to 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, and noting (3.4) that

$$\lim_{n\to\infty}(\lambda\alpha_n + \frac{1}{1-\beta^2}\langle f(q) - q, j(x_{n+1} - q)\rangle) \le 0.$$

Therefore applying Lemma 2.6 to (3.4), we conclude that $x_n \to q$.
The proof is complete. □

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