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A Note on Feng Qi Type Integral Inequalities

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Abstract

In this short note, by introducing parameters α, β, some sufficient conditions such that Qi type integral inequality

$$\int_a^b [f(x)]^\alpha dx \geq \left(\int_a^b f(x)dx\right)^\beta$$

hold are given, some new results are deduced.

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In [1], the following Qi type inequality was proved: If $f \in C^1[a,b]$, $f(a) \geq 0$, $f'(x) \geq (t-2)(x-a)^{t-3}$ for $x \in [a,b]$, and $t \geq 3$.Then

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)dx\right)^{t-1}. \tag{1}$$

If $t = n+2$. Then the following Qi integral inequality in [2] is obtained:

$$\int_a^b [f(x)]^{n+2} dx \geq \left(\int_a^b f(x)dx\right)^{n+1}. \tag{2}$$

In recent years, many valuable results (see [3-7]) have been obtained. In the short note, by introducing parameters α, β, some sufficient conditions such that Qi type integral inequality

$$\int_a^b [f(x)]^\alpha dx \geq \left(\int_a^b f(x)dx\right)^\beta. \tag{3}$$

hold are given. The main result is:

Theorem 0.1 *Suppose* $\alpha > \beta \geq 2$, $m = [\beta]$, $f(x) \in C^1[a,b]$, $f'(x) \geq f(x) \geq 0$ *and* $[f^{\alpha-\beta}(x)]' \geq (\alpha-\beta)\frac{\beta(\beta-1)\cdots(\beta-m+1)}{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)}(x-a)^{\beta-m}$. *Then*

$$\int_a^b [f(x)]^\alpha dx \geq \left(\int_a^b f(x)dx\right)^\beta. \tag{4}$$

Where $[\beta]$ *denote the integer part of* β.

Proof Sine $f'(x) \geq 0$, thus $f(x)$ is a increasing function on $[a,b]$. Let

$$F(x) = \int_a^x [f(u)]^\alpha du - \left(\int_a^x f(u)du\right)^\beta. \qquad x \in [a,b]$$

Then $F(a) = 0$, and we have

$$F'(x) = [f(x)]^\alpha - \beta\left(\int_a^x f(u)du\right)^{\beta-1} f(x)$$
$$= f(x)\left\{[f(x)]^{\alpha-1} - \beta\left(\int_a^x f(u)du\right)^{\beta-1}\right\} \triangleq f(x)F_1(x),$$

where

$$F_1(x) = [f(x)]^{\alpha-1} - \beta\left(\int_a^x f(u)du\right)^{\beta-1}.$$

Obviously, $F_1(a) = [f(a)]^{\alpha-1} \geq 0$ and

$$F_1'(x) = (\alpha-1)[f(x)]^{\alpha-2}f'(x) - \beta(\beta-1)\left(\int_a^x f(u)du\right)^{\beta-2} f(x)$$
$$\geq (\alpha-1)[f(x)]^{\alpha-1} - \beta(\beta-1)\left(\int_a^x f(u)du\right)^{\beta-2} f(x)$$
$$= f(x)\left\{(\alpha-1)[f(x)]^{\alpha-2} - \beta(\beta-1)\left(\int_a^x f(u)du\right)^{\beta-2}\right\} \triangleq f(x)F_2(x),$$

where

$$F_2(x) = (\alpha-1)[f(x)]^{\alpha-2} - \beta(\beta-1)\left(\int_a^x f(u)du\right)^{\beta-2}.$$

Obviously, $F_2(a) = (\alpha-1)[f(a)]^{\alpha-2} \geq 0$ and

$$F_2'(x) = (\alpha-1)(\alpha-2)[f(x)]^{\alpha-3}f'(x) - \beta(\beta-1)(\beta-2)\left(\int_a^x f(u)du\right)^{\beta-3} f(x)$$

$$\begin{aligned}
&\geq (\alpha-1)(\alpha-2)[f(x)]^{\alpha-2}-\beta(\beta-1)(\beta-2)\left(\int_a^x f(u)du\right)^{\beta-3} f(x)\\
&= f(x)\left\{(\alpha-1)(\alpha-2)[f(x)]^{\alpha-3}-\beta(\beta-1)(\beta-2)\left(\int_a^x f(u)du\right)^{\beta-3}\right\}\\
&\triangleq f(x)F_3(x).
\end{aligned}$$

Finally, we can obtain

$$\begin{aligned}
F_{m-1}(x) &= (\alpha-1)(\alpha-2)\cdots(\alpha-m+2)[f(x)]^{\alpha-m+1}\\
&\quad -\beta(\beta-1)(\beta-2)\cdots(\beta-m+2)\left(\int_a^x f(u)du\right)^{\beta-m+1}.
\end{aligned}$$

Obviously, $F_{m-1}(a)=(\alpha-1)\cdots(\alpha-m+2)[f(a)]^{\alpha-m+1}\geq 0$ and

$$\begin{aligned}
F'_{m-1}(x) &= (\alpha-1)(\alpha-2)\cdots(\alpha-m+1)[f(x)]^{\alpha-m}f'(x)\\
&\quad -\beta(\beta-1)(\beta-2)\cdots(\beta-m+1)\left(\int_a^x f(u)du\right)^{\beta-m} f(x)\\
&\geq (\alpha-1)(\alpha-2)\cdots(\alpha-m+1)[f(x)]^{\alpha-m}f'(x)\\
&\quad -\beta(\beta-1)(\beta-2)\cdots(\beta-m+1)[f(x)(x-a)]^{\beta-m}f(x)\\
&= [f(x)]^{\beta-m+1}\left\{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)[f(x)]^{\alpha-\beta-1}f'(x)\right\}\\
&\quad -[f(x)]^{\beta-m+1}\left\{\beta(\beta-1)(\beta-2)\cdots(\beta-m+1)(x-a)^{\beta-m}\right\}\\
&= [f(x)]^{\beta-m+1}\left\{\frac{1}{\alpha-\beta}(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)[f^{\alpha-\beta}(x)]'\right\}\\
&\quad -[f(x)]^{\beta-m+1}\left\{\beta(\beta-1)(\beta-2)\cdots(\beta-m+1)(x-a)^{\beta-m}\right\}.
\end{aligned}$$

Since

$$[f^{\alpha-\beta}(x)]' \geq (\alpha-\beta)\frac{\beta(\beta-1)\cdots(\beta-m+1)}{(\alpha-1)(\alpha-2)\cdots(\alpha-m+1)}(x-a)^{\beta-m},$$

thus $F'_{m-1}(x)\geq 0$ for $x\in[a,b]$, it follows that $F_{m-1}(x)$ is increasing on $[a,b]$, hence $F_{m-1}(x)\geq F_{m-1}(a)\geq 0$, it follows that $F'_{m-2}(x)\geq 0$, thus $F_{m-2}(x)$ is increasing on $[a,+\infty)$, $F_{m-2}(x)\geq F_{m-2}(a)\geq 0$. we can obtain $F_1(x)\geq 0$, it follows that $F'(x)\geq 0$, thus $F(x)$ is increasing on $[a,b]$, hence $F(b)\geq F(a)\geq 0$, i.e.

$$\int_a^b [f(x)]^\alpha dx \geq \left(\int_a^b f(x)dx\right)^\beta.$$

Corollary 0.2 *Suppose $t\geq 3$, $m=[t-1]$, $f(x)\in C^1[a,b]$ on the interval $[a,b]$, $f'(x)\geq f(x)\geq 0$ and $f'(x)\geq (t-m)(x-a)^{t-m-1}$. Then*

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)dx\right)^{t-1}. \tag{5}$$

Proof Setting $\alpha = t$, $\beta = t - 1$ in (4).

Corollary 0.3 *Suppose* $n, k \in Z_+$, $k \geq 2$, $f(x) \in C^1[a, b]$, $f'(x) \geq f(x) \geq 0$ *and* $[f^k(x)]' \geq k\frac{k!n!}{(k+n-1)!}$. *Then*

$$\int_a^b [f(x)]^{n+k} dx \geq \left(\int_a^b f(x) dx \right)^n . \tag{6}$$

Proof Setting $\alpha = n + k$, $\beta = k$ in (4).

Corollary 0.4 *Suppose* $n \in Z_+$, $f(x) \in C^1[a, b]$ *on the interval* $[a, b]$, $f'(x) \geq f(x) \geq 0$ *and* $f'(x) \geq 1$. *Then*

$$\int_a^b [f(x)]^{n+2} dx \geq \left(\int_a^b f(x) dx \right)^{n+1} .$$

Proof Setting $\alpha = n + 2$, $\beta = n + 1$ in (4).

Example: Suppose $n \in Z_+$, $f(x) = e^x$. Then $f'(x) = e^x = f(x) > 0$ and $f'(x) = e^x \geq 1$ for $x \in [0, 1]$. By Corollary 0.2, we obtain

$$\int_0^1 (e^x)^{n+2} dx \geq \left(\int_0^1 e^x dx \right)^{n+1} ,$$

it follows that

$$e^{n+2} - 1 \geq (n + 2)(e - 1)^{n+1}, \qquad n \in Z_+.$$

Remark: If $n \geq 3$, $t = n + 2$, $f(x) = e^x (x \in [0, 1])$, then $f(x)$ doesn't satisfy the condition in [1]: $f'(x) \geq (t - 2)(x - a)^{t-3}$ i.e. $e^x \geq nx^{n-1}$; f(x) doesn't also satisfy the condition in [2]: $f^{(n)}(x) \geq n!$ i.e. $e^x \geq n!$. Thus (8) not be obtained by the theorems in [1] and [2].

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