

On Almost Sure Convergence for Dependent Stochastic Sequence¹

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Abstract. In this paper, we obtain some limit theorems for concerning local convergence of arbitrary adapted stochastic sequence. The key tools are the stopping technique and martingale convergence theorem.

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1. INTRODUCTION

Let $\{\xi_n, \mathcal{F}_n, n \geq 1\}$ be a adapted sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$, namely, $\{\mathcal{F}_n, n \geq 1\}$ is an increasing sequence of σ -fields with $\mathcal{F}_n \subset \mathcal{F}$ for each $n \geq 1$ and $\mathcal{F}_0 = \{\phi, \Omega\}$, ξ_n is \mathcal{F}_n -measurable for each $n \geq 1$.

The limit behavior of random sequence is a classical subject in probability theory and statistics. For its importance and wide applications it has been investigated by many authors and attracted extension attentions (see ref. [1]-[4]). In the present paper, we shall study $\sum_{k=1}^n \frac{\xi_k}{a_k}$ converges almost surely under various Chung's type conditions. The key tools are the stopping technique and martingale convergence theorem.

2. MAIN RESULTS AND PROOFS

In this section, before presenting all the main results, we first give two lemmas, they will be needed in this section. Denote by $I_{(A)}$ the indicator function of a set A . Also in this section, c denotes a finite positive constant which may be different at different place.

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Lemma 1. Let $\{\xi_n, n \geq 1\}$ be an arbitrary integrable random variables, τ is a stopping time, if $\sum_{k=1}^n \xi_k$ a.s. convergence, then $\sum_{k=1}^{\tau \wedge n} \xi_k$ a.s. convergence.

Proof. Since $\sum_{k=1}^{\tau \wedge n} \xi_k = \sum_{i=1}^{n-1} \sum_{k=1}^i \xi_k I_{(\tau=i)} + \sum_{k=1}^n \xi_k I_{(\tau \geq n)}$ It is easy to see that $\xi_{\tau \wedge n}$ converges a.s. to $\sum_{k=1}^i \xi_k$ on set $\{\tau = i\}$ and $\sum_{k=1}^n \xi_k$ converges a.s. can imply $\sum_{k=1}^{\tau \wedge n} \xi_k$ converges a.s. □

Lemma 2. (Lévy – Doob,[4]) Let $\{\xi_n, \mathcal{F}_n, n \geq 1\}$ be a martingale difference sequence, then $\sum_{k=1}^{\infty} E[\xi_k^2 | \mathcal{F}_{k-1}] < \infty$ a.s. implies $\sum_{k=1}^n \xi_k$ converges a.s.

In the further consideration we also need the following concept of the equivalence in the sense of Khinchin.

Definition 1. The sequence $\{\xi_n, n \geq 0\}$ and $\{\eta_n, n \geq 1\}$ are said to be equivalent in the sense of Khinchin if $\sum_{n=1}^{\infty} P[\xi_n \neq \eta_n] < \infty$.

Theorem 1. Let $\{\xi_n, n \geq 1\}$ be given as above and $\{a_n, n \geq 1\}$ be a sequence of positive numbers, $\{\psi_n(x), n \geq 1\}$ be a sequence of Borel functions: $\psi_n : (0, +\infty) \rightarrow (0, +\infty)$ verifying that, for $0 \leq \alpha_n \leq \beta_n \leq 1, n \geq 1$ exist constants $C_n, D_n > 0$, such that if

$$u \geq v > 0 \implies C_n \frac{u^{\alpha_n}}{v^{\alpha_n}} \leq \frac{\psi_n(u)}{\psi_n(v)} \leq D_n \frac{u^{\beta_n}}{v^{\beta_n}}, \tag{2.1}$$

Let

$$\mathcal{L} := \left\{ \omega : \sum_{n=1}^{\infty} A_n E\left[\frac{\psi_n(|\xi_n|)}{\psi_n(a_n)} \mid \mathcal{F}_{n-1}\right] < \infty \right\} \tag{2.2}$$

where $A_n = \max\{\frac{1}{C_n}, D_n\}$. Then

$$\sum_{n=1}^{\infty} \frac{\xi_n}{a_n} \text{ converges a.s. on } \omega \in \mathcal{L} \tag{2.3}$$

Proof. Let $\eta_n = \xi_n I_{(|\xi_n| \leq a_n)}$, let l be a positive integral number, and

$$B_l := \left\{ \omega : \sum_{n=1}^{\infty} A_n E\left[\frac{\psi_n(|\xi_n|)}{\psi_n(a_n)} \mid \mathcal{F}_{n-1}\right] \leq l \right\} \tag{2.4}$$

$$\tau_l := \min\left\{ n : \sum_{k=1}^{n+1} A_k E\left[\frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} \mid \mathcal{F}_{k-1}\right] > l \right\} \tag{2.5}$$

where $\tau_l = \infty$, if the right-hand side of (2.5) is empty. It is easy to see that $\sum_{k=1}^{\tau_l \wedge n} A_k \frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} = \sum_{k=1}^n A_k \frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} I_{(\tau_l \geq k)}$, $I_{(\tau_l \geq k)}$ is \mathcal{F}_{k-1} measurable. Furthermore

$$E\left[\sum_{k=1}^{\tau_l \wedge n} A_k \frac{\psi_k(|\xi_k|)}{\psi_k(a_k)}\right] = E\left[\sum_{k=1}^n A_k \frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} I_{(\tau_l \geq k)}\right]$$

$$\begin{aligned}
 &= E\left\{\sum_{k=1}^n A_k E\left[\frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} I_{(\tau_l \geq k)} \mid \mathcal{F}_{k-1}\right]\right\} \\
 &= E\left\{\sum_{k=1}^n A_k I_{(\tau_l \geq k)} E\left[\frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} \mid \mathcal{F}_{k-1}\right]\right\} \\
 &= E\left\{\sum_{k=1}^{\tau_l \wedge n} A_k E\left[\frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} \mid \mathcal{F}_{k-1}\right]\right\} \leq l.
 \end{aligned} \tag{2.6}$$

Since $B_l = \{\tau_l = \infty\}$, we have from (2.6) that, for all n

$$\begin{aligned}
 &\sum_{k=1}^n A_k \int_{B_l} \frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} dP = \sum_{k=1}^n A_k E\left[\frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} I_{B_l}\right] \\
 &= E\left[I_{B_l} \sum_{k=1}^n A_k \frac{\psi_k(|\xi_k|)}{\psi_k(a_k)}\right] = E\left[I_{(\tau_l = \infty)} \sum_{k=1}^n A_k \frac{\psi_k(|\xi_k|)}{\psi_k(a_k)}\right] \\
 &= E\left[I_{(\tau_l = \infty)} \sum_{k=1}^{\tau_l \wedge n} A_k \frac{\psi_k(|\xi_k|)}{\psi_k(a_k)}\right] \leq \sum_{k=1}^{\tau_l \wedge n} A_k E\frac{\psi_k(|\xi_k|)}{\psi_k(a_k)} \leq l.
 \end{aligned} \tag{2.7}$$

Hence

$$\sum_{n=1}^{\infty} A_n \int_{B_l} \frac{\psi_n(|\xi_n|)}{\psi_n(a_n)} dP \leq l. \tag{2.8}$$

which implies that

$$\begin{aligned}
 &\sum_{n=1}^{\infty} P[B_l(\xi_n \neq \eta_n)] = \sum_{n=1}^{\infty} \int_{B_l(|x_n| > a_n)} dP \leq \sum_{n=1}^{\infty} \int_{B_l(|x_n| > a_n)} A_n \frac{|x_n|^{\alpha_n}}{a_n^{\alpha_n}} dP \\
 &\leq \sum_{n=1}^{\infty} \int_{B_l(|x_n| > a_n)} A_n \frac{\psi_n(|x_n|)}{\psi_n(a_n)} dP \leq \sum_{n=1}^{\infty} A_n \int_{B_l} \frac{\psi_n(|x_n|)}{\psi_n(a_n)} dP \leq l
 \end{aligned} \tag{2.9}$$

Hence $\{\xi_n, n \geq 1\}$ $\{\eta_n, n \geq 1\}$ are equivalent in the sense of Khinchin. i.e.

$$P[B_l(\xi_n \neq \eta_n), i.o.] = 0 \tag{2.10}$$

By the Borel-Cantelli lemma, it holds that

$$\sum_{n=1}^{\infty} \frac{\xi_n - \eta_n}{a_n} \text{ converges a.s. on } B_l \tag{2.11}$$

Since $\mathcal{L} = \bigcup_{l=1}^{\infty} B_l$, so we get further

$$\sum_{n=1}^{\infty} \frac{\xi_n - \eta_n}{a_n} \text{ converges a.s. on } \mathcal{L} \tag{2.12}$$

From condition (2.1), we have

$$\frac{|x|}{a_n} \leq \frac{|x|^{\beta_n}}{a_n^{\beta_n}} \leq A_n \frac{\psi_n(|x|)}{\psi_n(a_n)}, \text{ for } |x| \leq a_n$$

Note that

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_{B_l(|x_n| \leq a_n)} \frac{|x_n|}{a_n} dP \leq \sum_{n=1}^{\infty} \int_{B_l(|x_n| \leq a_n)} A_n \frac{\psi_n(|x_n|)}{\psi_n(a_n)} dP \\ & \leq \sum_{n=1}^{\infty} A_n \int_{B_l} \frac{\psi_n(|x_n|)}{\psi_n(a_n)} dP \leq l \end{aligned} \tag{2.13}$$

Hence $\sum_{n=1}^{\infty} \frac{\eta_n}{a_n}$ converges absolutely on B_l , and so it converges *a.s.* on B_l . Since $\mathcal{L} = \bigcup_l B_l$, we get the desired result.

$$\sum_{n=1}^{\infty} \frac{\eta_n}{a_n} \text{ converges a.s. on } \mathcal{L} \tag{2.14}$$

Hence the assertion of the theorem 1 follows immediately from (2.12) and (2.14). □

Theorem 2. *Let $\{\xi_n, n \geq 1\}$ be an arbitrary integrable stochastic sequence, let $\{a_n, n \geq 1\}$ be a sequence of positive real predictable random variables such that $\sum_{n=1}^{\infty} a_n < \infty, a.s.$. Let $\{\phi_n(x), n \geq 1\}$ be a sequence of Borel functions: $\phi_n : (0, +\infty) \rightarrow (0, +\infty)$ verifying that, for some $r_n \geq 2$ there exists constant $D > 0$, such that*

$$u \geq v > 0 \implies D \frac{u^{r_n}}{v^{r_n}} \leq \frac{\phi_n(u)}{\phi_n(v)}, \tag{2.15}$$

Let

$$\mathcal{J} := \left\{ \omega : \sum_{n=1}^{\infty} \frac{a_n}{\phi(\sqrt{a_n})} E[\phi_n(|\xi_n|) | \mathcal{F}_{n-1}] < \infty \right\} \tag{2.16}$$

then

$$\sum_{n=1}^{\infty} [\xi_n - E(\xi_n | \mathcal{F}_{n-1})] \text{ converges a.s. } \omega \in \mathcal{J} \tag{2.17}$$

Proof. In order to investigate the martingale limit theorem (2.17), we define random variables ζ_n by $\zeta_n = \xi_n - E(\xi_n | \mathcal{F}_{n-1})$, hence $\{\zeta_n, \mathcal{F}_n\}$ is a martingale difference sequence (*i.e.*, $\{\sum_{k=1}^n \zeta_k, \mathcal{F}_n, n \geq 1\}$ is a martingale), and let d be a positive real number, define

$$\rho := \inf \left\{ n \geq 1 : \sum_{k=1}^{n+1} \frac{a_k}{\phi_k(\sqrt{a_k})} E[\phi_k(|\xi_k|) | \mathcal{F}_{k-1}] > d \right\}, \tag{2.18}$$

Thus ρ is a stopping time and $\{\sum_{k=1}^{n \wedge \rho} \zeta_k = \sum_{k=1}^{n \wedge \rho} \zeta_k I_{(\rho \geq k)}, \mathcal{F}_{\rho \wedge n}, n \geq 1\}$ is a martingale

$$\begin{aligned} E\left|\sum_{k=1}^{n \wedge \rho} \zeta_k\right|^2 &= E\left|\sum_{k=1}^{n \wedge \rho} \zeta_k I_{(\rho \geq k)}\right|^2 \leq cE\sum_{k=1}^n |\zeta_k|^2 I_{(\rho \geq k)} \\ &= c\sum_{k=1}^n E\{E[|\zeta_k|^2 I_{(\rho \geq k)} | \mathcal{F}_{k-1}]\} = cE\left\{\sum_{k=1}^n E[|\zeta_k|^2 I_{(\rho \geq k)} | \mathcal{F}_{k-1}]\right\} \\ &= cE\left[\sum_{k=1}^{n \wedge \rho} E[|\zeta_k|^2 | \mathcal{F}_{k-1}]\right] \end{aligned} \tag{2.19}$$

We have by the conditional expectation property and Hölder's inequality.

$$E[|\zeta_k|^2 | \mathcal{F}_{k-1}] \leq E^{2/r_k}[|\zeta_k|^{r_k} | \mathcal{F}_{k-1}] \leq cE^{2/r_k}[|\xi_k|^{r_k} | \mathcal{F}_{k-1}] \tag{2.20}$$

Note that

$$E^{2/r_k}[|\zeta_k|^{r_k} | \mathcal{F}_{k-1}] = E^{2/r_k}\{[|\xi_k|^{r_k} I_{(|\xi_k| \leq \sqrt{a_k})} + |\xi_k|^{r_k} I_{(|\xi_k| > \sqrt{a_k})}] | \mathcal{F}_{k-1}\} \tag{2.21}$$

If $|\xi_k| > \sqrt{a_k}$, condition (2.15) and (2.21) imply

$$\left[\frac{|\xi_k|}{\sqrt{a_k}}\right]^{r_k} \leq \frac{1}{D} \frac{\phi_n(|\xi_k|)}{\phi_n(\sqrt{a_k})}$$

i.e.

$$|\xi_k|^{r_k} \leq \frac{\sqrt{a_k}}{D} \frac{\phi_n(|\xi_k|)}{\phi_n(\sqrt{a_k})} \tag{2.22}$$

(2.21) together with (2.22) yield

$$E^{2/r_k}[|\xi_k|^{r_k} | \mathcal{F}_{k-1}] \leq a_k + \frac{a_k}{D} E\left[\frac{\phi_k(|\xi_k|)}{\phi_k(\sqrt{a_k})} | \mathcal{F}_{k-1}\right] \tag{2.23}$$

Since $\sum_{n=1}^\infty a_n < \infty$ a.s., we have by (2.23)

$$E\left|\sum_{k=1}^{\rho \wedge n} \zeta_k\right|^2 \leq cE\left\{\sum_{k=1}^{\rho \wedge n} a_k + \frac{a_k}{D} \sum_{k=1}^{\rho \wedge n} E\left[\frac{\phi_k(|\xi_k|)}{\phi_k(\sqrt{a_k})} | \mathcal{F}_{k-1}\right]\right\} \leq c\left[\sum_{k=1}^\infty a_k + d\right] \tag{2.24}$$

i.e. $\sum_{k=1}^{\rho \wedge n} \zeta_k$ is L_2 bounded, thus

$$\sum_{k=1}^{\rho \wedge n} \zeta_k \text{ converges a.s. on } \omega \in \mathcal{J}. \tag{2.25}$$

Note that

$$\left\{\sum_{k=1}^\infty a_k E\left[\frac{\phi_k(|\xi_k|)}{\phi_k(\sqrt{a_k})} | \mathcal{F}_{k-1}\right] \leq d\right\} = (\rho = +\infty) \tag{2.26}$$

$$\sum_{k=1}^\infty \zeta_k I_{(\rho \geq k)} = \sum_{k=1}^\infty \zeta_k. \tag{2.27}$$

These and lemma 1 imply Theorem 2. □

Corollary 1. (see ref.[4]) Let $\{\xi_n, n \geq 1\}$ satisfy the conditions of Theorem 2. If

$$\mathcal{J} := \left\{ \omega : \sum_{n=1}^{\infty} a_n^{1-r_n/2} E[|\xi_n|^{r_n} | \mathcal{F}_{n-1}] < \infty \right\} \tag{2.28}$$

then

$$\sum_{n=1}^{\infty} [\xi_n - E(\xi_n | \mathcal{F}_{n-1})] \text{ converges a.s. } \omega \in \mathcal{J} \tag{2.29}$$

Proof. Letting $\phi_n(x) = x^{r_n}, n \geq 1$ in Theorem 2, it follows. □

Theorem 3. Let $\{\xi_n, \mathcal{F}_n, n \geq 1\}$ be a sequence of adapted random variables and let $\varphi : \mathbb{R} \rightarrow \mathbb{R}^+$ be even, continuous and nondecreasing on $(0, +\infty)$ function with $\lim_{x \rightarrow \infty} \varphi(x) = \infty$, and such that

$$\frac{\varphi(x)}{x} \downarrow \tag{2.30}$$

or

$$\frac{\varphi(x)}{x} \uparrow \text{ and } \frac{\varphi(x)}{x^p} \downarrow \text{ as } x \rightarrow \infty \tag{2.31}$$

for some $1 < p \leq 2$, suppose that

$$\sum_{k=1}^{\infty} [E \frac{\varphi^r(\xi_k)}{\varphi^r(a_k) + \varphi^r(\xi_k)} | \mathcal{F}_{k-1}] < \infty \text{ a.s.} \tag{2.32}$$

for $r = 1$ in case (2.30) and $r = p$ in case (2.31), where $\{a_n, n \geq 1\}$ is a sequence of positive numbers. Then

$$\sum_{k=1}^n \frac{\xi_k}{a_k} \text{ a.s. (P) converges iff } \sum_{k=1}^n E[\frac{\xi_k}{a_k} | \mathcal{F}_{k-1}] \text{ a.s. (P) converges} \tag{2.33}$$

Proof. Define

$$\begin{aligned} G_n &:= \sum_{k=1}^n [E(I_{(|\xi_k| \geq a_k)} | \mathcal{F}_{k-1}) - I_{(|\xi_k| \geq a_k)}], \\ K_n &:= \sum_{k=1}^n E[\frac{|\xi_k|}{a_k} I_{(|\xi_k| > a_k)} | \mathcal{F}_{k-1}], \quad K_{\infty} := \lim_{n \rightarrow \infty} K_n, \\ L_n &:= \sum_{k=1}^n E[\frac{|\xi_k|}{a_k} I_{(|\xi_k| \geq a_k)}], \quad L_{\infty} := \lim_{n \rightarrow \infty} L_n \end{aligned}$$

It is easily seen that $\{G_n, \mathcal{F}_n, n \geq 1\}$ is a martingale such that $E \sup_{n \geq 1} |G_n - G_{n-1}| \leq 1$.

Since

$$\sum_{n=1}^{\infty} [E(I_{(|\xi_n| \geq a_n)} | \mathcal{F}_{n-1})]$$

$$\leq 2 \sum_{n=1}^{\infty} [E \frac{\varphi^r(\xi_n)}{\varphi^r(a_n) + \varphi^r(\xi_n)} | \mathcal{F}_{n-1}] < \infty \tag{2.34}$$

Hence $\sup_{n \geq 1} G_n < \infty$ and G_n converges almost surely (see ref.[1]), and

$$\sum_{n=1}^{\infty} [I_{(|\xi_n| \geq a_n)}] < \infty. \tag{2.35}$$

i.e. $P(|\xi_n| \geq a_n \text{ i.o.}) = 0$ which imply that L_n a.s. converges

Assume $\lambda, \mu > 0$, define

$$\sigma := \inf\{n \geq 1, L_{n+1} > \lambda\}, \tau := \inf\{n \geq 1, \sum_{k=1}^{n+1} E[\frac{\varphi^r(\xi_k)}{\varphi^r(a_k) + \varphi^r(\xi_k)} | \mathcal{F}_{k-1}] \geq \mu\}$$

Obviously σ, τ are stopping times, and

$$E(K_{n \wedge \sigma}) = \sum_{k=1}^n E[\frac{|\xi_k|}{a_k} I_{(|\xi_k| \geq a_k) \cap (\sigma \geq k)}] = E(L_{n \wedge \sigma}) \leq \lambda.$$

Note that, for any $n \geq 1$ $K_{n \wedge \sigma} = K_n$ on $\{\sigma = \infty\}$, we obtain

$$E(K_{n \wedge \sigma} I_{(\sigma = \infty)}) = E(K_n I_{(\sigma = \infty)}) \leq \lambda.$$

We have by the monotonic theorem

$$E[K_{\infty} I_{(\sigma = \infty)}] = \lim_{n \rightarrow \infty} E(K_n I_{(\sigma = \infty)}) \leq \lambda$$

Hence

$$K_{\infty} I_{(\sigma = \infty)} < \infty$$

In view of the arbitrariness of λ , we get $K_{\infty} < \infty$, thus

$$K_n = \sum_{k=1}^n E[\frac{|\xi_k|}{a_k} I_{(|\xi_k| \geq a_k)} | \mathcal{F}_{k-1}] \text{ a.s. converges.} \tag{2.36}$$

Note that

$$\begin{aligned} & \sum_{n=1}^{\infty} E[\frac{\varphi^r(\xi_n)}{\varphi^r(a_n) + \varphi^r(\xi_n)} I_{(\tau \geq n)}] \\ &= \sum_{n=1}^{\infty} E[E[\frac{\varphi^r(\xi_n)}{\varphi^r(a_n) + \varphi^r(\xi_n)} | \mathcal{F}_{n-1}] I_{(\tau \geq n)}] \\ &\leq \sum_{k=1}^{\infty} E[\sum_{n=1}^k E[\frac{\varphi^r(\xi_n)}{\varphi^r(a_n) + \varphi^r(\xi_n)} | \mathcal{F}_{n-1}] I_{(\tau = n)}] \\ &\leq \sum_{k=1}^{\infty} E[\mu I_{(\tau = n)}] \leq \mu \end{aligned} \tag{2.37}$$

Moreover, for $n \geq 1$, let $H_n := \sum_{k=1}^n \frac{\xi_k}{a_k}$, then $H_{n \wedge \tau} = \sum_{k=1}^n \frac{\xi_k}{a_k} I_{(\tau \geq k)}$, define $M_1 := 0$, $M_n := \sum_{k=2}^n E[\frac{\xi_k}{a_k} | \mathcal{F}_{k-1}] (n \geq 2)$

If $\{M_n, n \geq 1\}$ *a.s.*(P) converges, note that

$$M_{n \wedge \tau} = \sum_{i=2}^{n-1} \sum_{k=2}^i E[\frac{\xi_k}{a_k} | \mathcal{F}_{k-1}] I_{(\tau=i)} + \sum_{k=2}^n E[\frac{\xi_k}{a_k} | \mathcal{F}_{k-1}] I_{(\tau \geq n)} (n \geq 2)$$

Thus

$$M_{n \wedge \tau} \text{ converges to } \sum_{k=2}^i E[\frac{\xi_k}{a_k} | \mathcal{F}_{k-1}] \text{ on set } \{\tau = i\}$$

and

$$M_n \text{ a.s.}(P) \text{ converges} \Rightarrow M_{n \wedge \tau} \text{ a.s.}(P) \text{ converges on set } \{\tau = \infty\}$$

so $M_{n \wedge \tau}$ *a.s.*(P) converges. Similarly, $K_{n \wedge \tau}$ and $L_{n \wedge \tau}$ *a.s.*(P) converges. Let

$$U_n := \frac{\xi_n}{a_n} I_{(|\xi_n| < a_n) \cap (\tau \geq n)} - E[\frac{\xi_n}{a_n} I_{(|\xi_n| < a_n) \cap (\tau \geq n)} | \mathcal{F}_{n-1}]$$

It is easy to see that $\{\sum_{k=1}^n U_k, \mathcal{F}_n\}$ is a martingale.

In order to prove $H_{n \wedge \tau}$ *a.s.*(P) converges, noting that $H_{n \wedge \tau} = L_{n \wedge \tau} - K_{n \wedge \tau} + \sum_{k=1}^n U_k + M_{n \wedge \tau}$, we only need to prove $\sum_{k=1}^n U_k$ *a.s.*(P) converges.

In fact,

$$\begin{aligned} E|\sum_{k=1}^n U_k|^2 &\leq c \sum_{k=1}^n E U_k^2 \\ &\leq c \sum_{k=1}^n \frac{\xi_k^2}{a_k^2} I_{(|\xi_k| < a_k) \cap (\tau \geq k)} \\ &\leq c \sum_{k=1}^n E \frac{\varphi^r(\xi_k)}{\varphi^r(a_k) + \varphi^r(\xi_k)} I_{(|\xi_k| < a_k) \cap (\tau \geq k)} \\ &\leq c \sum_{k=1}^n E \frac{\varphi^r(\xi_k)}{\varphi^r(a_k) + \varphi^r(\xi_k)} I_{(\tau \geq k)} \end{aligned}$$

Thus $\sum_{k=1}^n U_k$ converges(L^2), so $\sum_{k=1}^n U_k$, *a.s.*(P) converges which imply that $H_{n \wedge \tau}$ *a.s.*(P) converges. Noting that $H_{n \wedge \tau} = H_n$ on set $\{\tau = \infty\}$ this together with the arbitrariness of μ imply H_n *a.s.*(P) converges.

On the other hand, if H_n *a.s.*(P) converges, then $H_{n \wedge \tau}$ *a.s.*(P) converges. By noticing that $M_{n \wedge \tau} = H_{n \wedge \tau} - L_{n \wedge \tau} + K_{n \wedge \tau} - \sum_{k=1}^n U_k$ *a.s.*(P) converges, similar as in the proof of the above, the theorem 3 follows immediately. \square

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