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Multiplication and Composition Operators on Orlicz-Lorentz Spaces

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Abstract. Invertible and compact multiplication operators on Orlicz-Lorentz spaces $L_{\varphi,w}$ are characterized. Boundedness of the composition transformation C_T on $L_{\varphi,w}$ is also characterized.

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1 Introduction

Let f be a complex-valued measurable function defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$. For $s \geq 0$, define μ_f the *distribution function* of f as

$$\mu_f(s) = \mu\{x \in X : |f(x)| > s\}.$$

By f^* we mean the *non-increasing rearrangement* of f given as

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}, \quad t \geq 0.$$

By a weight function [6] w, we mean $w : (0,\infty) \to (0,\infty)$ is a non-increasing locally integrable function such that $\int_0^\infty w(t)dt = \infty$.

Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous convex function such that

(i) $\varphi(x) = 0 \iff x = 0$,

(ii) $\lim_{x\to\infty} \varphi(x) = \infty$.

Such a function is known as a Young function. The Young function φ is said to satisfy the Δ_2-condition if for some $k > 0$,

$$\varphi(2x) \leq k\varphi(x), \quad \forall \ x > 0.$$

If φ satisfy Δ_2-condition, then the class of simple functions is dense in $L_{\varphi,w}$ where $L_{\varphi,w}$ is defined as

$$\Big\{f : X \to \mathbf{C} \ \texttt{measurable} : \int_0^\infty \varphi\Big(\alpha f^*(t)\Big)w(t)dt < \infty, \ \texttt{for some} \ \alpha > 0\Big\}.$$

The space $L_{\varphi,w}$ is called an *Orlicz-Lorentz space* and is a Banach space with respect to the Luxemberg norm

$$||f||_{\varphi,w} = \inf\Big\{\epsilon > 0 : \int_0^\infty \varphi\Big(\frac{|f^*(t)|}{\epsilon}\Big)w(t)dt \leq 1\Big\}.$$

For more details on Orlicz-Lorentz spaces one can refer [6], [12] and the references therein.

Let $F(X)$ be a function space on a non-empty set X. Let $u : X \to \mathbf{C}$ be a function such that $u \cdot f \in F(X)$ whenever $f \in F(X)$. Then the transformation $f \mapsto u \cdot f$ on $F(X)$ is denoted by M_u. In case $F(X)$ is a topological space and M_u is continuous, we call it a multiplication operator induced by u.

If in a measure space X, Y is a measurable subset of X and $T : Y \to X$ is a measurable transformation, then we define the linear transformation C_T from $L_{\varphi,w}$ into the space of all complex-valued measurable functions on X as

$$(C_T f)(x) = \begin{cases} f(T(x)), & \text{if } x \in Y; \\ 0, & \text{otherwise} \end{cases}$$

for all $f \in L_{\varphi,w}$. If C_T is bounded with range in $L_{\varphi,w}$ we say that C_T is a composition operator on $L_{\varphi,w}$ induced by T.

For systematic study of the multiplication operators on different spaces we refer to [1, 2, 3, 8, 15 and 18] and for the study of composition operators on different function spaces we refer to [5, 9, 10, 13, 15 and 16].

2 Multiplication operators

In this section boundedness and invertibility of the multiplication operator M_u are characterized in terms of the boundedness and invertibility of the complex-valued measurable function u respectively.

Theorem 2.1 *The linear transformation $M_u : f \mapsto u \cdot f$ on the Orlicz-Lorentz space $L_{\varphi,w}$ is bounded if and only if u is essentially bounded. Moreover $||M_u|| = ||u||_\infty$.*

Proof. Let $u \in L_\infty(\mu)$, then we find

$$(u \cdot f)^*(t) \leq ||u||_\infty f^*(t).$$

Then

$$\int_0^\infty \varphi\Big(\frac{(M_u f)^*(t)}{||u||_\infty ||f||_{\varphi,w}}\Big) w(t)dt \ \leq \ \int_0^\infty \varphi\Big(\frac{||u||_\infty f^*(t)}{||u||_\infty ||f||_{\varphi,w}}\Big) w(t)dt \ \leq 1.$$

Hence for $f \in L_{\varphi,w}$,

$$||M_u f||_{\varphi,w} \ \leq \ ||u||_\infty ||f||_{\varphi,w}. \tag{1}$$

Conversely, suppose M_u is a bounded operator. If u is not essentially bounded function, then for every $n \in \mathbf{N}$, the set

$E_n = \{x \in X : |u(x)| > n\}$ has positive measure. Now

$$\chi_E^*(t) \ = \ \chi_{[0,\ \mu(E_n))}(t)$$

and

$$(u\chi_{E_n})^*(t) \ \geq \ n\chi_{E_n}^*(t).$$

This gives us

$$\begin{aligned} ||M_u E_n||_{\varphi,w} \ &\geq \ \inf\Big\{\lambda > 0 : \int_0^\infty \varphi\Big(\frac{n\chi_{E_n}^*(t)}{\lambda}\Big) w(t)dt \leq 1\Big\} \\ &= \ n||\chi_{E_n}||_{\varphi,w}. \end{aligned}$$

This contradicts the boundedness of M_u. Hence u must be essentially bounded.

Clearly from (1) we obtain $||M_u|| \ \leq \ ||u||_\infty$. For $\epsilon > 0$, let E denote the set $\{x \in X : |u(x)| \geq ||u||_\infty - \epsilon\}$ having positive measure. Then

$$\int_0^\infty \varphi\Big(\frac{(||u||_\infty - \epsilon)\chi_E^*(t)}{||M_u\chi_E||_{\varphi,w}}\Big) w(t)dt \ \leq \int_0^\infty \varphi\Big(\frac{(u\chi_E)^*(t)}{||M_u\chi_E||_{\varphi,w}}\Big) w(t)dt \ \leq 1.$$

Hence

$$||\chi_E||_{\varphi,w} \ \leq \ \frac{||M_u\chi_E||_{\varphi,w}}{||u||_\infty - \epsilon}$$

which proves that $||M_u|| \ \geq \ ||u||_\infty - \epsilon$. This gives $||M_u|| \ = \ ||u||_\infty$.

Theorem 2.2 *The set of all multiplication operators on $L_{\varphi,w}$ is a maximal abelian subalgebra of $\mathcal{B}(L_{\varphi,w})$, the algebra of all bounded operators on $L_{\varphi,w}$.*

Proof. The proof follows on the similar lines as in case of Lorentz space [2].

Corollary 2.3 *The multiplication operator M_u on $L_{\varphi,w}$ is invertible if and only if u is invertible in $L_\infty(\mu)$.*

Theorem 2.4 *Let $M_u \in \mathcal{B}(L_{\varphi,w})$. Then M_u is compact if and only if $L_{\varphi,w}((u,\epsilon))$ is finite dimensional for each $\epsilon > 0$, where*

$$(u,\epsilon) = \{x \in X : |u(x)| \geq \epsilon\} \quad \textit{and} \quad L_{\varphi,w}((u,\epsilon)) = \{f\chi_{(u,\epsilon)} : f \in L_{\varphi,w}\}.$$

Proof. If M_u is a compact, then $L_{\varphi,w}((u,\epsilon))$ is a closed invariant subspace of M_u and hence $M_u|_{L_{\varphi,w}((u,\epsilon))}$ is a compact operator. Moreover for each $f \in L_{\varphi,w}$,

$$\|M_u f\chi_{(u,\epsilon)}\|_{\varphi,w} \ \geq \ \epsilon\|f\chi_{(u,\epsilon)}\|_{\varphi,w}.$$

Thus $M_u|_{L_{\varphi,w}((u,\epsilon))}$ has closed range in $L_{\varphi,w}((u,\epsilon))$ and hence invertible. Being compact, $L_{\varphi,w}((u,\epsilon))$ is finite dimensional.

Conversely, suppose that $L_{\varphi,w}((u,\epsilon))$ is finite dimensional for each $\epsilon > 0$. In particular for each natural number n, $L_{\varphi,w}((u,1/n))$ is finite dimensional, then for each n, define $u_n : X \to \mathbf{C}$ as

$$u_n(x) = \begin{cases} u(x), & \text{x} \in \text{(u, 1/n)}, \\ 0, & \text{otherwise.} \end{cases}$$

We find that

$$((u_n - u)\cdot f)^*(t) \ \leq \ \frac{1}{n} f^*(t), \quad \forall \ \ t > 0$$

Consequently

$$\|M_{u_n} f - M_u f\|_{\varphi,w} \leq \ \frac{1}{n} \ \|f\|_{\varphi,w},$$

which implies that M_{u_n} converges to M_u uniformly. As $L_{\varphi,w}((u,\epsilon))$ is finite dimensional so M_{u_n} is a finite rank operator. Therefore M_{u_n} is a compact operator and hence M_u is a compact operator.

Corollary 2.5 *If $(X,\mathcal{A},\mu)$ is a non-atomic measure space, then the only compact multiplication operator from $L_{\varphi,w}$ into itself is the zero operator.*

Theorem 2.6 *Let $M_u \in \mathcal{B}(L_{\varphi,w})$. Then M_u has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e.(μ) on $S = \{x \in X : u(x) \neq 0\}$, the support of u.*

Proof. If $|u(x)| \geq \delta$ $a.e.(\mu)$ on S, then for $f \in L_{\varphi,w},\ t > 0$

$$(uf\chi_S)^*(t) \ \geq \ \delta(f\chi_S)^*(t)$$

and so

$$\|M_u f\chi_S\|_{\varphi,w} \geq \ \delta\|f\chi_S\|_{\varphi,w}.$$

Hence M_u has closed range.

Conversely if M_u has closed range, then there exists an $\epsilon > 0$ such that

$$\|M_u f\|_{\varphi,w} \ \geq \ \epsilon\|f\|_{\varphi,w},$$

for all $f \in L_{\varphi,w}(S)$, where $L_{\varphi,w}(S) = \{f\chi_S : f \in L_{\varphi,w}\}$.

Let $E = \{x \in S : |u(x)| < \epsilon/2\}$. If $\mu(E) > 0$, then we can find a measurable set $F \subseteq E$ such that $\chi_F \in L_{\varphi,w}(S)$. It is known that

$$||\chi_F||_{\varphi,w} \ = \ \frac{1}{\varphi^{-1}(\frac{1}{k})}$$

where $k \ = \ \int_0^{\mu(F)} w(t)dt$. Also we have for $t > 0$,

$$\left\{\frac{\epsilon}{2}\ s > 0 : \mu_{\chi_F,w}(s) \leq t\right\} \subseteq \left\{s > 0 : \mu_{u\cdot\chi_F,w}(s) \leq t\right\}$$

so that

$$(u \cdot \chi_F)^*(t) \leq \frac{\epsilon}{2}\ (\chi_F)^*(t).$$

Hence

$$\begin{aligned}
\|M_u\chi_F\|_{\varphi,w} \ &= \ \inf\left\{\epsilon > 0 : \int_0^\infty \varphi\Big(\frac{(u\chi_F)^*(t)}{\epsilon}\Big)w(t)dt\right\} \\
&< \ \inf\left\{\epsilon > 0 : \int_0^\infty \varphi\Big(\frac{\frac{\epsilon}{2}\chi_F^*(t)}{\epsilon}\Big)w(t)dt\right\} \\
&= \ \frac{\epsilon}{2}\ \|\chi_F\|_{\varphi,w},
\end{aligned}$$

which is a contradiction. Therefore $\mu(E) = 0$. This completes the proof.

3 Composition operators

In this section a necessary and sufficient condition for the boundedness of composition mapping C_T is given. If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space and $T : X \to X$ is a non-singular measurable transformation and w is a weight function, define a measure ν on the σ-algebra $\mathcal{A}$ as

$$\nu(E) \ = \ \int_0^{\mu(E)} w(t)dt.$$

Then $\nu \ll \mu$ and w is the Radon-Nikodym derivative of ν with respect to μ. Also $\nu T^{-1} \ll \nu$. Now for $E \in \mathcal{A}$,

$$\begin{aligned} ||\chi_E||_{\varphi,w} &= \inf\Big\{k > 0 : \int_0^{\infty} \varphi\Big(\frac{\chi_E^*(t)}{k}\Big) w(t)dt \ \leq \ 1\Big\} \\ &= \inf\Big\{k > 0 : \int_0^{\mu(E)} \varphi\Big(\frac{1}{k}\Big) w(t)dt \ \leq \ 1\Big\} \\ &= \frac{1}{\varphi^{-1}\Big(\frac{1}{\nu(E)}\Big)}. \end{aligned}$$

Theorem 3.1 *Let $T : X \to X$ be a non-singular measurable transformation Then C_T $(f \mapsto f \circ T)$ induced by T is bounded on $L_{\varphi,w}$ if and only if there exists a constant $M \geq 1$ such that*

$$\nu T^{-1}(E) \ \leq \ M\nu(E), \quad \forall \ \ E \in \mathcal{A}. \tag{2}$$

Proof. Let C_T is a composition operator, then we can find $M \geq 1$ such that

$$||C_T f||_{\varphi,w} \ \leq \ M||f||_{\varphi,w}, \quad \forall \ \ f \in L_{\varphi,w}.$$

If $E \in \mathcal{A}$ is such that $\nu(E) = \infty$, then (2) holds. Suppose $E \in \mathcal{A}$ is such that $\nu(E) < \infty$. Then $\chi_E \in L_{\varphi,w}$ and hence

$$||C_T \chi_E||_{\varphi,w} \ \leq \ M||\chi_E||_{\varphi,w}.$$

This implies

$$\frac{1}{\varphi^{-1}\Big(\frac{1}{\nu(T^{-1}(E))}\Big)} \ \leq \ M \frac{1}{\varphi^{-1}\Big(\frac{1}{\nu(E)}\Big)},$$

that is

$$\varphi^{-1}\Big(\frac{1}{\nu(E)}\Big) \ \leq \ M\varphi^{-1}\Big(\frac{1}{\nu(T^{-1}(E))}\Big),$$

Using [14] (φ^{-1} is concave) we have

$$\frac{1}{\nu(E)} \ \leq \ M \frac{1}{\nu(T^{-1}(E))},$$

or

$$\nu(T^{-1}(E)) \ \leq \ M\nu(E).$$

Conversely, if inequality (2) holds for all $E \in \mathcal{A}$, then

$$\int_0^{\mu T^{-1}(E)} w(t)dt \ \leq \ \int_0^{M\mu(E)} w(t)dt$$

for all $E \in \mathcal{A}$. This implies

$$\nu(T^{-1}(E)) \leq M\nu(E), \quad \forall \;\; E \in \mathcal{A}$$

and therefore for $t > 0$

$$(f \circ T)^*(Mt) \;\leq\; f^*(t).$$

Since $\varphi(\alpha t) \;\leq\; \alpha\varphi(t)$ for $\alpha < 1$ and w is non-increasing

$$\begin{aligned}\int_0^\infty \varphi\Big(\frac{(f\circ T)^*(t)}{M||f||_{\varphi,w}}\Big)w(t)dt \quad &\leq \quad \int_0^\infty M\varphi\Big(\frac{f^*(t)}{M||f||_{\varphi,w}}\Big)w(t)dt \\ &\leq \quad \int_0^\infty \varphi\Big(\frac{f^*(t)}{M||f||_{\varphi,w}}\Big)w(t)dt \;\leq 1.\end{aligned}$$

Hence

$$||C_T f||_{\varphi,w} \;\leq\; M||f||_{\varphi,w}, \quad \forall \;\; f \in L_{\varphi,w}.$$

This gives the boundedness of C_T.

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