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Strong Convergence of Monotone Hybrid Algorithm for Quasi-Nonexpansive Mappings

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Abstract. The purpose of this article is to prove strong convergence theorems of fixed point for quasi-nonexpansive mappings. In order to get the strong convergence theorems for quasi-nonexpansive mappings, the monotone hybrid iteration method is presented and is used to approximate the fixed point of quasi-nonexpansive mappings. Noting that, the hybrid iteration method presented by S.Matsushita and W.Takahashi can be used for nonexpansive mapping but it can not be used for quasi-nonexpansive mappings. However, the monotone hybrid method can be used for quasi-nonexpansive mappings. The results of this paper modify and improve the results of S.Matsushita and W.Takahashi and some others.

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1. Introduction

Let K be a nonempty closed convex subset of a real Banach space E. A mapping $T : K \to E$ is called quasi-nonexpansive [1-4] if

$$\|p - Tx\| \leq \|p - x\|, \ \ \forall\, p \in F(T),\ \forall\, x \in K$$

where $F(T) = \{x \in K : x = Tx\}$ is the fixed points set of T, and it is assumed that $F(T)$ be nonempty. T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \ \forall\, x, y \in K.$$

It is clear that a nonexpansive mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. However, there exist quasi-nonexpansive mappings that are not nonexpansive. Let $T : R \to R$ be defined by $Tx = \frac{x}{2}\sin\frac{1}{x}$ if $x \neq 0$ and $T0 = 0$. Then T is quasi-nonexpansive but not nonexpansive (see [5]). The quasi-nonexpansive mappings has been introduced and studied by various authors[1-4], but the results of this paper is very different to them.

In recent years, the hybrid (CQ) iteration methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors[6–9].

In 2003, Nakajo and Takahashi[6] proposed the following modification of Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.1}$$

where C is a closed convex subset of H, P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.1) converges strongly to $P_{F(T)}(x_0)$. Where $F(T)$ denote the fixed points set of T.

In 2006, T.H.Kim and H.K.Xu[7] proposed the following modification of the Mann iteration method for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.2}$$

where C is bounded closed convex subset and

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0 \quad as\ n \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$.

They also proposed the following modification of the Mann iteration method for asymptotically nonexpansive semigroup $\Im$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \overline{\theta}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where C is bounded closed convex subset and

$$\overline{\theta}_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(u)du)^2 - 1](diam C)^2 \to 0 \ \ as \ n \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(\Im)}(x_0)$. Where $F(\Im)$ denote the common fixed points set of $\Im$.

In 2006, Carlos Martinez-Yanes and Hong-Kun Xu[8] proposed the following modification of the Ishikawa iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad +(1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where C is a closed convex subset of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one and $\beta_n \to 0$, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}(x_0)$.

Carlos Martinez-Yanes and Hong-Kun Xu[8] proposed also the following modification of the Halpern iteration method for nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ \qquad +\alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.5}$$

where C is a closed convex subset of H. They proved that if the sequence $\alpha_n \to 0$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}(x_0)$.

The purpose of this article is to prove strong convergence theorems of fixed point for quasi-nonexpansive mappings. In order to get the strong convergence theorems for quasi-nonexpansive mappings, the monotone hybrid iteration method is presented and is used to approximate the fixed point of quasi-nonexpansive mappings. Noting that, the hybrid iteration method presented by S.Matsushita and W.Takahashi can be used for nonexpansive mapping but it can not be used for quasi-nonexpansive mappings. However, the monotone hybrid method can be used for quasi-nonexpansive mappings. The results of this paper modify and improve the results of S.Matsushita and W.Takahashi and some others.

2. Strong convergence for quasi-nonexpansive mappings

Theorem 2.1. *Let E be a Hilbert space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a closed quasi-nonexpansive mapping with nonempty fixed points set $F(T)$. Assume that $\{\alpha_n\}$ is a sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \ \ \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \|z - y_n\| \le \|z - x_n\|\}, \\ C_0 = \{z \in C : \|z - y_0\| \le \|z - x_0\|\}, \\ Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ Q_0 = C, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{2.1}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.

Proof. Let H be a real Hilbert space, C be a any closed convex subset of H. Given points $x, y \in H$, we claim that, the set

$$D = \{v \in C : \|y - v\| \le \|x - v\|\}$$

is closed convex. As a matter of fact, the defining inequality in D is equivalent to the inequality

$$\langle 2(x-y), v\rangle \le \|x\|^2 - \|y\|^2.$$

This inequality is affine in v and hence the set D is convex. This implies that, C_0 is closed convex. Since $Q_0 = C$ is closed convex, so that C_1 and Q_1 are closed convex. Therefore, by induction, we obtain that, C_n and Q_n are closed convex for all $n \ge 0$.

Next, we show that the fixed points set $F(T) \subset C_n$ for all $n \ge 0$, indeed, we have, for all $p \in F(T)$,

$$\|y_n - p\|^2 = \|\alpha_n(x_n - p) + (1-\alpha_n)(Tx_n - p)\|^2$$

$$\leq \alpha_n\|x_n - p)\|^2 + (1-\alpha_n)\|x_n - p\|^2$$
$$= \|x_n - p\|^2$$

So $p \in C_n$ for every $n \geq 0$, therefore, $F(T) \subset C_n$ for every $n \geq 0$.

Next, we show that $F(T) \subset C_n \bigcap Q_n$ for all $n \geq 0$. It suffices to show that $F(T) \subset Q_n$, for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0, \quad \forall z \in Q_n \cap C_n,$$

as $F(T) \subset C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$. Hence the $F(T) \subset C_n \bigcap Q_n$ holds for all $n \geq 0$.

By Lemma 2.1, $F(T)$ is closed convex, there exists a unique element $z_0 \in F(T)$ such that $z_0 = P_{F(T)}(x_0)$. From $x_{n+1} = P_{C_n \bigcap Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in C_n \bigcap Q_n$. As $z_0 \in F(T) \subset C_n \bigcap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

Next, we show that $\{x_n\}$ is Cauchy sequence, hence $\{x_n\}$ converges strongly to a point $p \in C$. As $x_{n+1} = P_{C_n \bigcap Q_n}(x_0) \subset Q_n$ and $x_n = P_{Q_n}(x_0)$, we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with Lemma 2.3 implies that, there exists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_0 - x_n\rangle \geq 0$ and hence

$$\begin{aligned} \|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \ n \to \infty \end{aligned} \tag{2.2}$$

for any $m \geq 1$. So that $\{x_n\}$ is a Cauchy sequence in C, then there exists a point $p \in C$ such that $\lim_{n\to\infty} x_n = p$.

Finally, we show that, $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}(x_0)$. Since $x_{n+1} \in C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2$$

therefore

$$\|y_n - x_{n+1}\| \to 0.$$

This combined with (2.2) implies that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0.$$

Noticing that $Tx_n = y_n - \alpha_n(x_n - Tx_n)$, we have

$$\|x_n - Tx_n\| \leq \|x_n - y_n\| + \alpha_n\|x_n - Tx_n\|.$$

It follows that

$$\|x_n - Tx_n\| \leq \frac{1}{1-\alpha_n}\|x_n - y_n\|.$$

Since $\limsup_{n\to\infty} \alpha_n < 1$, then we obtain

$$\|x_n - Tx_n\| \to 0.$$

We have proved that $\{x_n\}$ converges in norm to a point $p \in C$ which together with the $\|Tx_n - x_n\| \to 0$ implies that p is a fixed point of T.

We claim that $p = z_0 = P_{F(T)}(x_0)$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exists a positive integer N, if $n > N$ then $\|x_0 - x_n\| > \|x_0 - z_0\|$ which leads to

$$\|z_0 - x_0\|^2 = \|z_0 - x_n + x_n - x_0\|^2$$

$$= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$ which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F(T)$, this is a contradiction. This complete the proof.

Corollary 2.2. *Let E be a Hilbert space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a nonexpansive mapping with nonempty fixed points set $F(T)$. Assume that $\{\alpha_n\}$ is a sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \|z - y_n\| \leq \|z - x_n\|\}, \\ C_0 = \{z \in C : \|z - y_0\| \leq \|z - x_0\|\}, \\ Q_n = \{z \in C_{n-1} \bigcap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.

Remark. In recent years, the hybrid iteration methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors[1-4]. In fact that, all hybrid iteration methods can be replaced (or modified) by monotone hybrid iteration methods respectively. On the other hand, by using the monotone hybrid method we can easy show the iteration sequence $\{x_n\}$ is Cauchy sequence, so that without use of the Kadec-Klee property, demiclosedness principle and Opial's condition or other about weak topological technologies.

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