Int. Journal of Math. Analysis, Vol. 1, 2007, no. 28, 1361 - 1375

Higher-Order Dynamic Delay Differential Equations on Time Scales

Qinfu Sun, Hua Su [1] and Shixia Luan

School of Mathematical Sciences, Qufu Normal University
Qufu Shandong, 273165, People's Republic of China

Abstract

Let **T** be a time scale. We study the existence of positive solutions for the higher-order p-Laplacian dynamic delay differential equations on time scales. By using the fixed-point index theory, the existence of positive solution and many positive solutions for nonlinear four-point singular boundary value problem with p-Laplacian operator are obtained.

Mathematics Subject Classification: 34B16, 39A10

Keywords: Time scale, Higher-order dynamic delay differential equations, positive solutions, fixed-point index theory

1 Introduction

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [12], and is a new area of still fairly theoretical exploration in mathematics. Boundary value problems for delay differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory (see [5,4]). In recent years, many authors have begun to pay attention to the study of boundary-value problems or with p-Laplacian equations or with p-Laplacian dynamic equations on time scales (see [1-3,6-14] and the references therein).

In [8], Sun and Li considered the existence of positive solution of the following dynamic equations on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(t, u(t)) = 0, t \in (0, T), \tag{1.1}$$

$$\beta u(0) - \gamma u^{\Delta}(0) = 0,\ \alpha u(\eta) = u(T), \tag{1.2}$$

---

[1] jnsuhua@163.com

where $\beta,\ \gamma \geq 0,\ \beta+\gamma > 0,\ \eta \in (0,\rho(T)),\ 0 < \alpha < T/\eta$. They obtained the existence of single and multiple positive solutions of the problem (1.1) and (1.2) by using fixed point theorem and Leggett-Williams fixed point theorem, respectively.

In [14], Wang discussed the following dynamic equation on time scales by using Avery-Peterson fixed theorem (see [11]):

$$(\phi_p(u'))' + q(t)f(t,u(t),u(t-1),u'(t)) = 0,\ t \in (0,1), \tag{1.3}$$

$$u(t) = \xi(t),\ -1 \leq t \leq 0,\ u(1) = 0, \tag{1.4}$$

and

$$u(t) = \xi(t),\ -1 \leq t \leq 0,\ u'(1) = 0, \tag{1.4$'$}$$

They obtained some results for the existence three positive solutions of the problem (1.3), (1.4) and (1.3), (1.4′), respectively.

However, there are not many concerning the p-Laplacian problems on time scales. Especially, for the singular multi-point boundary value problems for higher-order p-Laplacian dynamic delay differential equations on time scales, with the author's acknowledge, no one has studied the existence of positive solutions in this case.

Now, motivated by the results mentioned above, in this paper, we study the existence of positive solutions for the following nonlinear four-point singular boundary value problem with higher-order p-Laplacian dynamic delay differential equations operator on time scales (SBVP):

$$\left(\phi_p\left(u^{\Delta^{n-1}}(t)\right)\right)^{\nabla} + g(t)f\left(u(t),u(t-\tau),u^{\Delta}(t),\cdots,u^{\Delta^{n-2}}(t)\right) = 0,\ 0 < t < T, \tag{1.5}$$

$$\begin{cases} u(t) = \zeta(t), \quad -\tau \leq t \leq 0, \\ u^{\Delta^i}(0) = 0,\ 1 \leq i \leq n-3, \\ \alpha\phi_p\left(u^{\Delta^{n-2}}(0)\right) - \beta\phi_p\left(u^{\Delta^{n-1}}(\xi)\right) = 0, \\ \gamma\phi_p\left(u^{\Delta^{n-2}}(T)\right) + \delta\phi_p\left(u^{\Delta^{n-1}}(\eta)\right) = 0, \quad n \geq 3, \end{cases} \tag{1.6}$$

where $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ \phi_q = \phi_p^{-1}$, $\dfrac{1}{p}+\dfrac{1}{q} = 1$. $\xi,\ \eta \in (0,T)$, $\tau \in [0,T]$ is prescribed and $\xi < \eta$, $g : (0,T) \to [0,\infty)$, $\alpha > 0, \beta \geq 0, \gamma > 0, \delta \geq 0$.

Our main tool of this paper is the following fixed point index theory.

Theorem 1.1[1]. Suppose E is a real Banach space, $K \subset E$ is a cone, let $\Omega_r = \{u \in K : \|u\| \leq r\}$. Let operator $T : \Omega_r \longrightarrow K$ be completely continuous and satisfy $Tx \neq x,\ \forall\ x \in \partial\Omega_r$. Then

(i) If $\|Tx\| \leq \|x\|,\ \forall\ x \in \partial\Omega_r$, then $i(T,\Omega_r,K) = 1$;

(ii) If $\|Tx\| \geq \|x\|,\ \forall\ x \in \partial\Omega_r$, then $i(T,\Omega_r,K) = 0$.

This paper is organized as follows. In section 2, we present some preliminaries and lemmas that will be used to prove our main results. In section 3, we discuss the existence of solution of the systems (1.5), (1.6). In section 4, we give a examples as the application.

2 Preliminary Notes

For convenience, we list here the following definitions which are needed later. We begin by presenting some basic definitions which can be found in [1,2].

A time scale $\mathbf{T}$ is an arbitrary nonempty closed subset of real numbers R. The operators σ and ρ from $\mathbf{T}$ to $\mathbf{T}$,

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T},\ \rho(t) = \sup\{\tau \in \mathbf{T} \mid \tau < t\} \in \mathbf{T},$$

are called the forward jump operator and the backward jump operator, respectively.

Let $f : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$ (assume t is not left-scattered if $t = \sup \mathbf{T}$), then the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq |\sigma(t) - s|, \quad \text{for all } s \in U.$$

Similarly, for $t \in \mathbf{T}$ (assume t is not right-scattered if $t = \inf \mathbf{T}$), the nabla derivative of f at the point t is defined in [1] to be the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq |\rho(t) - s|, \quad \text{for all } s \in U.$$

If $\mathbf{T} = R$, then $x^{\Delta}(t) = x^{\nabla}(t) = x'(t)$. If $\mathbf{T} = Z$, then $x^{\Delta}(t) = x(t+1) - x(t)$ is the forward difference operator while $x^{\nabla}(t) = x(t) - x(t-1)$ is the backward difference operator.

A function f is left-dense continuous (i.e., ld-continuous), if f is continuous at each left-dense point in $\mathbf{T}$ and its right-sided limit exists at each right-dense point in $\mathbf{T}$. It is well-known that if f is ld-continuous, then there is a function $F(t)$ such that $F^{\nabla}(t) = f(t)$.

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Define $f^{\Delta^n}(t)$ to be the delta derivative of $f^{\Delta^{n-1}}(t)$, i.e., $f^{\Delta^n}(t) = (f^{\Delta^{n-1}}(t))^{\Delta}$.

In the rest of this article, $\mathbf{T}$ is closed subset of R with $0 \in \mathbf{T}_k, T \in \mathbf{T}^k$. And let

$$B = \left\{u \in C[-\tau, 0] \cap C_{ld}^{n-2}[0, T] : \ u^{\Delta^i}(0) = 0, \ 0 \le i \le n-3\right\}.$$

Then B is a Banach space with the norm $\|u\| = \max\limits_{t \in [0,T]} |u^{\Delta^{n-2}}(t)|$. And let

$$K = \left\{u \in B : \ u^{\Delta^{n-2}}(t) \ge 0, \ u^{\Delta^{n-2}}(t) \text{ is concave function}, t \in [0, T]\right\}.$$

Obviously, K is a cone in B. Set $K_r = \{u \in K : \|u\| \le r\}$.

In the rest of the paper, we also make the following assumptions:

(H_1) $f \in C_{ld}([0, +\infty)^n, [0, +\infty))$;

(H_2) $g(t) \in C_{ld}((0, T), [0, +\infty))$ and there exists $t_0 \in (0, T)$, such that

$$g(t_0) > 0, \ 0 < \int_0^T g(s) \, \nabla s < +\infty;$$

(H_3) $\zeta(t) \in C([-\tau, 0]$, $\zeta(t) > 0$ on $[-\tau, 0)$ and $\zeta(0) = 0$.

It is easy to check that condition (H_2) implies that

$$0 < \int_0^T \phi_q \left(\int_0^s g(s_1) \nabla s_1\right) \Delta s < +\infty.$$

We can easily get the following Lemmas.

Lemma 2.1 Suppose condition (H_2) holds. Then there exists a constant $\theta \in (0, \frac{1}{2})$ satisfies

$$0 < \int_\theta^{T-\theta} g(t) \nabla t < \infty.$$

Furthermore, the function

$$A(t) = \int_\theta^t \phi_q \left(\int_s^t g(s_1) \nabla s_1\right) \Delta s + \int_t^{T-\theta} \phi_q \left(\int_t^s g(s_1) \nabla s_1\right) \nabla s, \ t \in [\theta, T-\theta]$$

is positive continuous functions on $[\theta, T-\theta]$, therefore $A(t)$ has minimum on $[\theta, T-\theta]$. Hence we suppose that there exists $L > 0$ such that $A(t) \ge L, \ t \in [\theta, T-\theta]$.

Lemma 2.2 Let $u \in K$ and θ of Lemma 2.1, then

$$u(t) \ge \theta \|u\|, \ t \in [\theta, T-\theta].$$

The proof of the above two lemmas are similar to the proof of in [6, Lemma 2.1 and Lemma 2.2], so we omit it.

Lemma 2.3 Suppose that conditions (H_1), (H_2), (H_3) hold, $u(t) \in B \cap C_{ld}^{n-1}(0,1)$ is a solution of the following boundary value problems for $0 < t < T$,

$$\left(\phi_p\left(u^{\Delta^{n-1}}(t)\right)\right)' + \varphi(t) = 0, \tag{2.1}$$

$$\begin{cases} u(t) = 0, \quad -\tau \le t \le 0, \\ u^{\Delta^i}(0) = 0, \ 1 \le i \le n-3, \\ \alpha\phi_p\left(u^{\Delta^{n-2}}(0)\right) - \beta\phi_p\left(u^{\Delta^{n-1}}(\xi)\right) = 0, \\ \gamma\phi_p\left(u^{\Delta^{n-2}}(T)\right) + \delta\phi_p\left(u^{\Delta^{n-1}}(\eta)\right) = 0, \quad n \ge 3, \end{cases} \tag{2.2}$$

where

$$\varphi(t) = g(t)f\left(u(t), u(t-\tau) + h(t-\tau), u^{\Delta}(t), \cdots, u^{\Delta^{n-2}}(t)\right)$$

$$h(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0, \\ 0, & 0 \le t \le T. \end{cases}$$

Then, $\overline{u}(t) = u(t) + h(t)$, $-\tau \le t \le T$ is a positive solution to the SBVP (1.5), (1.6).

Proof It is easy to check that $\overline{u}(t)$ satisfies (1.5) and (1.6).

So in the rest section of this paper, we focus on SBVP (2.1) and (2.2).

Lemma 2.4 Suppose that conditions (H_1), (H_2), (H_3) hold, $u(t) \in B \cap C_{ld}^{n-1}(0,1)$ is a solution of boundary value problems (2.1), (2.2) if and only if $u(t) \in B$ is a solution of the following integral equation

$$u(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0, \\ \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2})\Delta s_{n-2}\Delta s_{n-3} \cdots \Delta s_1, & 0 \le t \le T, \end{cases}$$

where

$$w(t) = \begin{cases} \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma \varphi(s)\nabla s\right) + \int_0^t \phi_q\Big(\int_s^\sigma \varphi(r)\nabla r\Big)\Delta s, & 0 \le t \le \sigma, \\ \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta \varphi(s)\nabla s\right) + \int_t^T \phi_q\Big(\int_\sigma^s \varphi(r)\nabla r\Big)\Delta s, & \sigma \le t \le T. \end{cases} \tag{2.3}$$

Proof. *Necessity.*

Obviously, for $t \in (-\tau, 0)$, we have $u(t) = \zeta(t)$. If $t \in (0,1)$, by the equation of the boundary condition, we have $u^{\Delta^{n-1}}(\xi) \ge 0$, $u^{\Delta^{n-1}}(\eta) \le 0$, then there exist a constant $\sigma \in [\xi, \eta] \subset (0,T)$ such that $u^{\Delta^{n-1}}(\sigma) = 0$.

Firstly, by integrating the equation of the problems (2.1) on (σ, T), we have

$$\phi_p(u^{\Delta^{n-1}}(t)) = \phi_p(u^{\Delta^{n-1}}(\sigma)) - \int_\sigma^t \varphi(s)\nabla s, \tag{2.4}$$

then

$$u^{\Delta^{n-1}}(t) = -\phi_q\left(\int_\sigma^t \varphi(s)\nabla s\right),$$

thus

$$u^{\Delta^{n-2}}(t) = u^{\Delta^{n-2}}(\sigma) - \int_\sigma^t \phi_q\Big(\int_\sigma^s \varphi(r)\nabla r\Big)\Delta s. \tag{2.5}$$

By $u^{\Delta^{n-1}}(\sigma) = 0$ and condition (2.4), let $t = \eta$ on (2.4), we have

$$\phi_p(u^{\Delta^{n-1}}(\eta)) = -\int_\sigma^\eta \varphi(s)\nabla s.$$

By the equation of the boundary condition (2.2), we have

$$\phi_p(u^{\Delta^{n-2}}(T)) = -\frac{\delta}{\gamma}\phi_p(u^{\Delta^{n-1}}(\eta)),$$

then

$$u^{\Delta^{n-2}}(T) = \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta \varphi(s)\nabla s\right).$$

Then, by (2.5) and let let $t = T$ on (2.5), we have

$$u^{\Delta^{n-2}}(\sigma) = \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta \varphi(s)\nabla s\right) + \int_\sigma^T \phi_q\left(\int_\sigma^s \varphi(r)\nabla r\right)\Delta s. \tag{2.6}$$

Then

$$u^{\Delta^{n-2}}(t) = \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta \varphi(s)\nabla s\right) + \int_t^T \phi_q\left(\int_\sigma^s \varphi(r)\nabla r\right)\Delta s. \tag{2.7}$$

Then by integrating the equation (2.7) for $n-2$ times on $(0, T)$, we have

$$\begin{aligned} u(t) = &\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}} \phi_q\left(\frac{\sigma}{\gamma}\int_\delta^\eta \varphi(s)\nabla s\right)\Delta s_{s_{n-2}}\cdots\Delta s_2\Delta s_1 \\ &+\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\Big(\int_{s_{n-2}}^T \phi_q\left(\int_\sigma^s \varphi(r)\nabla r\right)\Delta s\Big)\Delta s_{s_{n-2}}\cdots\Delta s_2\Delta s_1. \end{aligned}$$

Similarly, for $t \in (0, \sigma)$, by integrating the equation of problems (2.1) on $(0, \sigma)$, we have

$$\begin{aligned} u(t) = &\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}} \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma \varphi(s)\nabla s\right)\Delta s_{s_{n-2}}\cdots\Delta s_2\Delta s_1 \\ &+\int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-3}}\Big(\int_0^{s_{n-2}} \phi_q\left(\int_s^\sigma \varphi(r)\nabla r\right)\Delta s\Big)\Delta s_{s_{n-2}}\cdots\Delta s_2\Delta s_1. \end{aligned}$$

Therefore, for any $t \in [0, T]$, $u(t)$ can be expressed as equation

$$u(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \displaystyle\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2}) \Delta s_{n-2} \Delta s_{n-3} \cdots \Delta s_1, & 0 \leq t \leq T, \end{cases}$$

where $w(t)$ is expressed as (2.3). Then the results of Lemma 2.2 holds.

Sufficiency.

Suppose that $u(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2}) \Delta s_{n-2} \Delta s_{n-3} \cdots \Delta s_1, 0 \leq t \leq T$. Then by (2.3), we have

$$u^{\Delta^{n-1}}(t) = \begin{cases} \phi_q\Big(\displaystyle\int_t^\sigma \varphi(s) \nabla s\Big) \geq 0, \ 0 \leq t \leq \sigma, \\ -\phi_q\Big(\displaystyle\int_\sigma^t \varphi(s) \nabla s\Big) \leq 0, \ \sigma \leq t \leq T, \end{cases} \tag{2.8}$$

So, $(\phi_p(u^{\Delta^{n-1}}))^\nabla + \varphi(t) = 0, \ 0 < t < T$. These imply that the equation (2.1) holds. Furthermore, by letting $t = 0$ and $t = T$ on (2.3) and (2.8), we can obtain the boundary value equations of (2.2). The proof is complete.

Now, we define a operator equation T given by

$$(Tu)(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \displaystyle\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} w(s_{n-2}) \Delta s_{n-2} \Delta s_{n-3} \cdots \Delta s_1, & 0 \leq t \leq T, \end{cases}$$

where $w(t)$ is given by (2.3).

From the definition of T and above discussion, we deduce that for each $u \in K$, $Tu \in K$. Moreover, we have the following Lemma.

Lemma 2.5 $T : K \to K$ is completely continuous.

Proof. Because

$$(Tu)^{\Delta^{n-1}}(t) = w^\Delta(t) = \begin{cases} \phi_q\left(\displaystyle\int_t^\sigma \varphi(s) \nabla s\right) \geq 0, & 0 \leq t \leq \sigma, \\ -\phi_q\left(\displaystyle\int_\sigma^t \varphi(s) \nabla s\right) \leq 0, & \sigma \leq t \leq T, \end{cases}$$

is continuous, decreasing on $[0, T]$ and satisfies $(Tu)^{\Delta^{n-1}}(\sigma) = 0$. Then, $Tu \in K$ for each $u \in K$ and $(Tu)^{\Delta^{n-2}}(\sigma) = \max_{t \in [0,T]} (Tu)^{\Delta^{n-2}}(t)$. This shows that $TK \subset K$. Furthermore, it is easy to check by Arzela-ascoli Theorem that $T : K \to K$ is completely continuous.

Lemma 2.6 Suppose that conditions $(H_1), \ (H_2), \ (H_3)$ hold, the solution $u(t)$ of problem (2.1), (2.2) satisfy:

$$\max_{0 \leq t \leq T} |u(t - \tau) + h(t - \tau)| \leq \max_{-\tau \leq t \leq 0} |\zeta(t)|,$$

$$u(t) \leq Tu^{\Delta}(t) \leq \cdots \leq T^{n-3}u^{\Delta^{n-3}}(t),\ t \in [0,T]$$

and for $\theta \in (0, \frac{1}{2})$ in Lemma 2.1, we have

$$u^{\Delta^{n-3}}(t) \leq \frac{T}{\theta}u^{\Delta^{n-2}}(t),\ t \in [\theta, T-\theta].$$

Proof. Firstly, we can have

$$\begin{aligned} \max_{0\leq t\leq T}|u(t-\tau)+h(t-\tau)| &\leq \max_{0\leq t\leq T}|u(t-\tau)| + \max_{0\leq t\leq T}|h(t-\tau)| \\ &= \max_{-\tau\leq t\leq T-\tau}|u(t)| + \max_{-\tau\leq t\leq T-\tau}|h(t)| \\ &= \max_{-\tau\leq t\leq 0}|\zeta(t)|. \end{aligned}$$

Next, if $u(t)$ is the solution of problem (2.1), (2.2), then $u^{\Delta^{n-2}}(t)$ is concave function, and $u^{\Delta^{i}}(t) \geq 0,\ i = 0,1,\cdots,n-2,\ t \in [0,T]$, Thus we have

$$u^{\Delta^{i}}(t) = \int_0^t u^{\Delta^{i+1}}(s)\Delta s \leq tu^{\Delta^{i+1}}(t) \leq Tu^{\Delta^{i+1}}(t),\ i = 0,1,\cdots,n-4,$$

i.e., $u(t) \leq Tu^{\Delta}(t) \leq \cdots \leq T^{n-3}u^{\Delta^{n-3}}(t),\ t \in [0,T]$.

Finally, by Lemma 2.2, for $t \in [\theta, T-\theta]$, we have $u^{\Delta^{n-2}}(t) \geq \theta\|u^{\Delta^{n-2}}\|$. By $u^{\Delta^{n-3}}(t) = \int_0^t u^{\Delta^{n-2}}(s)\Delta s \leq T\|u^{\Delta^{n-2}}\|$, we have

$$u^{\Delta^{n-3}}(t) \leq \frac{T}{\theta}u^{\Delta^{n-2}}(t),\ t \in [\theta, T-\theta].$$

The proof is complete.

For convenience, we set

$$H = \max_{-\tau\leq t\leq 0}|\zeta(t)|, \quad \theta^* = \frac{2}{L}, \quad \theta_* = \frac{1}{\left(T+\phi_q\left(\frac{\beta}{\alpha}\right)\right)\phi_q\left(\int_0^T g(r)\nabla r\right)}.$$

where L is the constant from Lemma 2.1. By Lemma 2.4, we can also set

$$f_0 = \lim_{u_n\to 0}\max_{(u_1,u_2,\cdots,u_n)\in\aleph}\frac{f(u_1,u_2,\cdots,u_n)}{u_n^{p-1}},$$

$$f_\infty = \lim_{u_n\to \infty}\min_{(u_1,u_2,\cdots,u_n)\in\aleph}\frac{f(u_1,u_2,\cdots,u_n)}{u_n^{p-1}},$$

where $\aleph = \{(u_1,u_2,\cdots,u_n)|\ 0 \leq u_1 \leq Tu_3 \cdots \leq T^{n-3}u_{n-1} \leq \frac{T^{n-2}}{\theta}u_n,\ u_2 \leq H\}$.

3 Main Results

In this section, we present our main results.

Theorem 3.1 Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy

(A_1): $f(u_1, u_2, \cdots, u_n) \geq (mr)^{p-1}$, for $\theta r \leq u_n \leq r$, $(u_1, u_2, \cdots, u_n) \in \aleph$;

(A_2): $f(u_1, u_2, \cdots, u_n) \leq (MR)^{p-1}$, for $0 \leq u_n \leq R$, $(u_1, u_2, \cdots, u_n) \in \aleph$,

where $m \in (\theta^*, \infty)$, $M \in (0, \theta_*)$.

Then, the (SBVP) (2.1), (2.2) has a solution u such that $\|u\|$ lies between r and R.

The proof of Theorem 3.1 Without loss of generality, we suppose that $r < R$. For any $u \in K$, by Lemma 2.2, we have

$$u^{\Delta^{n-2}}(t) \geq \theta\|u\|, \quad t \in [\theta, T-\theta]. \tag{3.1}$$

We define two open subset Ω_1 and Ω_2 of E:

$$\Omega_1 = \{u \in K : \|u\| < r\}, \ \Omega_2 = \{u \in K : \|u\| < R\}.$$

For any $u \in \partial\Omega_1$, by (3.1) we have

$$r = \|u\| \geq u^{\Delta^{n-2}}(t) \geq \theta\|u\| = \theta r, \quad t \in [\theta, T-\theta].$$

For $t \in [\theta, T-\theta]$ and $u \in \partial\Omega_1$, we shall discuss it from three perspectives.

(i) If $\sigma \in [\theta, T-\theta]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.3, we have

$$\begin{aligned} 2\|Tu\| &= 2(Tu)^{\Delta^{n-2}}(\sigma) \\ &\geq \int_0^\sigma \phi_q\left(\int_s^\sigma \varphi(r)\nabla r\right)\Delta s + \int_\sigma^T \phi_q\left(\int_\sigma^s \varphi(r)\nabla r\right)\Delta s \\ &\geq \int_\theta^\sigma \phi_q\left(\int_s^\sigma \varphi(r)\nabla r\right)\Delta s + \int_\sigma^{T-\theta} \phi_q\left(\int_\sigma^s \varphi(r)\nabla r\right)\Delta s \\ &\geq mrA(\sigma) \geq mrL \geq 2r = 2\|u\|. \end{aligned}$$

(ii) If $\sigma \in (T-\theta, T]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.3, we have

$$\begin{aligned} \|Tu\| &= (Tu)^{\Delta^{n-2}}(\sigma) \\ &\geq \ \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma \varphi(s)\nabla s\right) + \int_0^\sigma \phi_q\left(\int_s^\sigma \varphi(r)\nabla r\right)\Delta s \\ &\geq \int_\theta^{T-\theta} \phi_q\left(\int_s^{T-\theta} \varphi(r)\nabla r\right)\Delta s \\ &\geq mrA(T-\theta) \geq mrL \geq 2r > r = \|u\|. \end{aligned}$$

(iii) If $\sigma \in (0, \theta)$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.3, we have

$$\begin{aligned}\|Tu\| &= (Tu)^{\Delta^{n-2}}(\sigma)\\ &\geq \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta \varphi(s)\nabla s\right) + \int_\sigma^T \phi_q\left(\int_\sigma^s \varphi(r)\nabla r\right)\Delta s\\ &\geq \int_\theta^{T-\theta} \phi_q\left(\int_\theta^s \varphi(r)\nabla r\right)\Delta s\\ &\geq mrA(\theta) \geq mrL \geq 2r > r = \|u\|.\end{aligned}$$

Therefore, no matter under which condition, we all have

$$\|Tu\| \geq \|u\|, \ \forall\ u \in \partial\Omega_1.$$

Then by Theorem 1.1, we have

$$i(T, \Omega_1, K) = 0. \tag{3.2}$$

On the other hand, for $u \in \partial\Omega_2$, we have $u(t) \leq \|u\| = R$, by (A_2) we know

$$\begin{aligned}\|Tu\| &= (Tu)^{\Delta^{n-2}}(\sigma)\\ &\leq \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma \varphi(s)\nabla s\right) + \int_0^T \phi_q\left(\int_s^\sigma \varphi(r)\nabla r\right)\Delta s\\ &\leq \left(T + \phi_q\left(\frac{\beta}{\alpha}\right)\right) MR\phi_q\left(\int_0^T g(r)\nabla r\right) \leq R = \|u\|.\end{aligned}$$

thus

$$\|Tu\| \leq \|u\|, \forall\ u \in \partial\Omega_2.$$

Then, by Theorem 1.1, we have

$$i(T, \Omega_2, K) = 1. \tag{3.3}$$

Therefore, by (3.2), (3.3), $r < R$, we have

$$i(T, \Omega_2 \setminus \overline{\Omega}_1, K) = 1.$$

Then operator T has a fixed point $u \in (\Omega_1 \setminus \overline{\Omega}_2)$, and $r \leq \|u\| \leq R$. This completes the proof of Theorem 3.1.

Theorem 3.2 Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy

(A_3): $\quad f_0 = \varphi \in \left[0, \left(\frac{\theta_*}{4}\right)^{p-1}\right)$;

(A_4): $\quad f_\infty = \lambda \in \left(\left(\dfrac{2\theta^*}{\theta} \right)^{p-1}, \infty \right)$.

Then, the (SBVP) (2.1), (2.2) has a solution u such that $\|u\|$ lies between r and R.

The proof of Theorem 3.2

First, by $f_0 = \varphi \in \left[0, \left(\dfrac{\theta_*}{4} \right)^{p-1} \right)$, for $\epsilon = \left(\dfrac{\theta_*}{4} \right)^{p-1} - \varphi$, there exists an adequately small positive number ρ, as $0 \le u_n \le \rho,\ u_n \ne 0$, we have

$$f(u_1, u_2, \cdots, u_n) \le (\varphi + \epsilon)(u_n)^{p-1} \le \left(\frac{\theta_*}{4} \right)^{p-1} \rho^{p-1} = \left(\frac{\theta_*}{4} \rho \right)^{p-1}. \tag{3.4}$$

Then let $R = \rho,\ M = \dfrac{\theta_*}{4} \in (0, \theta_*)$, thus by (3.4)

$$f(u_1, u_2, \cdots, u_n) \le (MR)^{p-1},\ 0 \le u_n \le R.$$

So condition (A_2) holds.

Next, by condition (A_4), $f_\infty = \lambda \in \left(\left(\dfrac{2\theta^*}{\theta} \right)^{p-1}, \infty \right)$, then for $\epsilon = \lambda - \left(\dfrac{2\theta^*}{\theta} \right)^{p-1}$, there exists an appropriately big positive number $r \ne R$, as $u_n \ge \theta r$, we have

$$f(u_1, u_2, \cdots, u_n) \ge (\lambda - \epsilon)(u_n)^{p-1} \ge \left(\frac{2\theta^*}{\theta} \right)^{p-1} (\theta r)^{p-1} = (2\theta^* r)^{p-1}, \tag{3.5}$$

Let $m = 2\theta^* > \theta^*$, thus by (3.5), condition (A_1) holds. Therefor by Theorem 3.1 we know that the results of Theorem 3.2 holds. The proof of Theorem 3.2 is complete.

Theorem 3.3 Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy

(A_5): $\quad f_\infty = \lambda \in \left[0, \left(\dfrac{\theta_*}{4} \right)^{p-1} \right)$;

(A_6): $\quad f_0 = \varphi \in \left(\left(\dfrac{2\theta^*}{\theta} \right)^{p-1}, \infty \right)$.

Then, the (SBVP) (2.1), (2.2) has a solution u such that $\|u\|$ lies between r and R.

The proof of Theorem 3.3.

First, by condition (A_6), $f_0=\varphi\in\left(\left(\frac{2\theta^*}{\theta}\right)^{p-1},\infty\right)$, then for $\epsilon=\varphi-\left(\frac{2\theta^*}{\theta}\right)^{p-1}$, there exists an adequately small positive number r, as $0\le u_n\le r,\ u_n\ne 0$, we have

$$f(u_1,u_2,\cdots,u_n)\ge(\varphi-\epsilon)(u_n)^{p-1}=\left(\frac{2\theta^*}{\theta}\right)^{p-1}(u_n)^{p-1},$$

thus when $\theta r\le u_n\le r$, we have

$$f(u_1,u_2,\cdots,u_n)\ge\left(\frac{2\theta^*}{\theta}\right)^{p-1}(\theta r)^{p-1}=(2\theta^* r)^{p-1}. \tag{3.6}$$

Let $m=2\theta^*>\theta^*$, so by (3.6), condition (A_1) holds.

Next, by condition (A_5): $f_\infty=\lambda\in\left[0,\left(\frac{\theta_*}{4}\right)^{p-1}\right)$, then for $\epsilon=\left(\frac{\theta_*}{4}\right)^{p-1}-\lambda$, there exists an suitably big positive number $\rho\ne r$, as $u_n\ge\rho$, we have

$$f(u_1,u_2,\cdots,u_n)\le(\lambda+\epsilon)(u_n)^{p-1}\le\left(\frac{\theta_*}{4}\right)^{p-1}(u_n)^{p-1}. \tag{3.7}$$

If f is unbounded, by the continuity of f on $[0,\infty)^n$, then exists constant $R\ (\ne r\ )\ge\rho$, and a point $(u_{01},u_{02},\cdots,u_{0n})\in[0,\infty)^n$ such that

$$\rho\le u_{0n}\le R$$

and

$$f(u_1,u_2,\cdots,u_n)\le f(u_{01},u_{02},\cdots,u_{0n}),\qquad 0\le u_n\le R.$$

Thus, by $\rho\le u_{0n}\le R$, we know

$$f(u_1,u_2,\cdots,u_n)\le f(u_{01},u_{02},\cdots,u_{0n})\le\left(\frac{\theta_*}{4}\right)^{p-1}(u_{0n})^{p-1}\le\left(\frac{\theta_*}{4}R\right)^{p-1}.$$

Choose $M=\frac{\theta_*}{4}\in(0,\theta_*)$. Then, we have

$$f(u_1,u_2,\cdots,u_n)\le(MR)^{p-1},\ 0\le u_n\le R.$$

If f is bounded, we suppose $f(u_1,u_2,\cdots,u_n)\le\overline{M}^{p-1},\ u_n\in[0,\infty),\ \overline{M}\in R_+$, there exists an appropriately big positive number $R>\frac{4}{\theta_*}\overline{M}$, then choose $M=\frac{\theta_*}{4}\in(0,\theta_*)$, we have

$$f(u_1,u_2,\cdots,u_n)\le\overline{M}^{p-1}\le\left(\frac{\theta_*}{4}R\right)^{p-1}=(MR)^{p-1},\ 0\le u_n\le R.$$

Therefore, condition (A_2) holds. Therefore, by Theorem 3.1, we know that the results of Theorem 3.3 holds. The proof of Theorem 3.3 is complete.

4 Main Results

Example 4.1 Consider the following 3-order singular boundary value problem (SBVP) with p-Laplacian

$$\begin{cases} (\phi_p(u^{\Delta\Delta}))^{\nabla}(t)+\dfrac{1}{20}t^{-\frac{1}{2}}(u^{\Delta})^{\frac{1}{2}}(t)\cdot \\ \left[\dfrac{1}{5}+\dfrac{\frac{94}{5}e^{2u^{\Delta}(t)}}{120u(t)u(t-1)+7e^{u^{\Delta}(t)}+e^{2u^{\Delta}(t)}}\right]=0, \quad 0<t<1, \\ u(t)=t^2-1,\ -1\le t\le 0, \\ \phi_p(u^{\Delta}(0))-\phi_p(u^{\Delta\Delta}(\frac{1}{4}))=0,\ \phi_p(u^{\Delta}(1))+\delta\phi_p(u^{\Delta\Delta}(\frac{1}{2}))=0, \end{cases} \tag{5.1}$$

where

$$\alpha=\gamma=1,\ \beta=1,\ p=\frac{3}{2},\quad \delta\ge 0,\ \xi=\frac{1}{4},\quad \eta=\frac{1}{2},\ \theta=\frac{1}{4},\ \tau=T=1.$$

So, by Lemma 2.3, we discuss the following SBVP:

$$\begin{cases} (\phi_p(u^{\Delta\Delta}))^{\nabla}(t)+\dfrac{1}{20}t^{-\frac{1}{2}}(u^{\Delta})^{\frac{1}{2}}(t)\cdot \\ \left[\dfrac{1}{5}+\dfrac{\frac{94}{5}e^{2u^{\Delta}(t)}}{120u(t)[u(t-1)-h(t-1)]+7e^{u^{\Delta}(t)}+e^{2u^{\Delta}(t)}}\right]=0,\ 0<t<1, \\ u(t)=0,\ -1\le t\le 0, \\ \phi_p(u^{\Delta}(0))-\phi_p(u^{\Delta\Delta}(\frac{1}{4}))=0,\ \phi_p(u^{\Delta}(1))+\delta\phi_p(u^{\Delta\Delta}(\frac{1}{2}))=0, \end{cases} \tag{5.2}$$

where

$$h(t)=\begin{cases} t^2-1, & -1\le t\le 0, \\ 0, & 0\le t\le 1, \end{cases} \quad g(t)=\frac{1}{20}t^{-\frac{1}{2}},\quad \zeta(t)=t^2-1,$$

$$f(u_1,\ u_2,\ u_3)=(u_3)^{\frac{1}{2}}\left[\frac{1}{5}+\frac{\frac{94}{5}e^{2u_3}}{120u_1u_2+7e^{u_3}+e^{2u_3}}\right].$$

Then obviously, $q=3$,

$$H=\max_{-1\le t\le 0}|\zeta(t)|=1,\ f_0=\varphi=\lim_{u_3\to 0^+}\max_{0\le u_1\le\frac{1}{4}u_3,0\le u_2\le 1}\frac{f(u_1,\ u_2,\ u_3)}{u_3^{p-1}}=\frac{51}{20},$$

$$f_\infty = \lambda = \lim_{u_3\to\infty} \min_{0\le u_1\le \frac{1}{4}u_3, 0\le u_2\le 1} \frac{f(u_1,\ u_2,\ u_3)}{u_3^{p-1}} = \frac{95}{5},\quad \int_0^T g(t)\nabla t = \frac{1}{10},$$

so conditions (H_1), (H_2), (H_3) hold.

Next,

$$\theta_* = \frac{1}{\left(T+\phi_q\left(\frac{\beta}{\alpha}\right)\right)\phi_q\left(\int_0^T g(r)\nabla r\right)} = 50,$$

then $\left(\frac{\theta_*}{4}\right)^{p-1} = \frac{5\sqrt{2}}{2} > \frac{51}{20}$, i.e. $\varphi \in \left[0, \left(\frac{\theta_*}{4}\right)^{p-1}\right)$, so conditions (A_3) holds.

For $\theta = \frac{1}{4}$, it is easy see by calculating that

$$L = \min_{t\in[\theta, T-\theta]} A(t) = \frac{1}{16}\left(\frac{7}{36}+\frac{\sqrt{3}}{3}\right).$$

Because of

$$\left(\frac{2\theta^*}{\theta}\right)^{p-1} = 96\times\left(\frac{1}{7+12\sqrt{3}}\right)^{\frac{1}{2}} < \frac{95}{5},$$

then

$$\lambda \in \left(\left(\frac{2\theta^*}{\theta}\right)^{p-1}, \infty\right),$$

so conditions (A_4) holds. Then by Theorem 3.2, SBVP (4.2) has at least a positive solution $u(t)$. So, $\overline{u}(t) = u(t) + h(t)$, $-1 < t < 1$ is the positive solution of SBVP (4.1).

Acknowledgements

Research supported by the NNSF of China (10701049,10771117), Shandong University(306001) and the Doctoral Program Foundation of Education Ministry of China(20060446001).

References

[1] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cone. Academic Press, Sandiego, 1988.

[2] D. R. Anderson, Solutions to second-order three-point problems on time scales, J. Differ. Equ. Appl. 8 (2002) 673-688.

[3] E. R. Kaufmann, Positive solutions of a three-point boundary value problem on a time scale, Eletron. J. Diff. Equ. 82 (2003) 1-11.

[4] G. B. Gustaf, K. Schmitt, Nonzero solutions of boundary value problems for second-order ordinary and delay differential equations, J. Diff. Equ., 12(1972)129-147.

[5] L. H. Erbe, Q. K. Kong, boundary value Problems for singular second-order functional differential equations, J. Comput. Appl. Math. 53 (1994)377-388.

[6] H. Su, Z. Wei, F. Xu, The Existence of Positive Solutions for Nonlinear Singular Boundary Value System with p-Laplacian, J. Appl. Math. Comp. 181(2006)826-836.

[7] H. Su, Z. Wei, F. Xu, The Existence of Countably Many Positive Solutions For A System of Nonlinear Singular Boundary Value Problems With The p-Laplacian Operator, J. Math. Anal. Appl. 325(2007)319-332.

[8] H. R. Sun, W. T.Li, Positive solutions for nonlinear three-point boundary value problems on time scales, J. Math. Anal. Appl. 299 (2004) 508-524.

[9] H. Su, B. Wang, Z. Wei, Positive Solutions of Four-Point Boundary Value Problems for Four-order p-Laplacian dynamic equations on Time Scales, Elec. J. Diff. Equ., 78(2006)1-13.

[10] J. P. Sun, Existence of solution and positive solution of BVP for nonlinear third-order dynamic equation, Nonlinear Anal. 64(2006) 629-636.

[11] R. I. Avery and D. R. Anderson; Existence of three positive solutions to a second-order boundary value problem on a measure chain, J. Comput. Appl. Math. 141 (2002), 65-73.

[12] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, Results Math. 18(1990)18-56.

[13] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, Dynamic systems on measure chains, Kluwer Academic Publishers, boston, 1996.

[14] Y. Y. Wang, W. X. Zhao and W. G. Ge, Multiple positive solutions for boundary value problems of second order delay differential equations with one-dimensional p-Laplacian, J. Math. Anal. Appl. 326(2007)641-654.

Received: September 23, 2007