Int. Journal of Math. Analysis, Vol. 1, 2007, no. 28, 1389 - 1395

Some Fixed Point Theorems for Non Self Maps

Neeta Singh

Department of Mathematics
University of Allahabad
Allahabad-211002, India
n_s32132@yahoo.com

Abstract

We prove some fixed point theorems for condensing, compact and non-expansive non-self maps.

Mathematics Subject Classification: 47H09

Keywords: Nonexpansive mapping, condensing maps, compact mapping, Banach space

1 Introduction and preliminary definitions

Fixed point theorems have been a major theoretical tool in the fields of differential equations, topology, economics, game theory, dynamics, optical control and functional analysis. Moreover the usefulness of the concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points. The plan of the paper is as follows. In section 2 of this paper we prove a fixed point theorem for a condensing map defined on a closed subset of a Banach space E. In section 3 we have a fixed point theorem for a continuous compact map and another for the sum of two operators, one being compact and the other a k-set contractive map. This section also includes a theorem for a contractive map defined on a closed subset of a Banach space under various boundary conditions. The following theorem is given in [1].

Let U be a bounded, open, convex subset of a uniformly convex Banach space X with $0 \in U$ and $F : \overline{U} \to X$ a nonexpansive map. Then either,
(A_1) *F has a fixed point in $\overline{U}$ or*
(A_2) *There exist $\lambda \in (0,1)$ and $u \in \partial U$ with $u = \lambda F(u)$*
is true.

In section 4 of this paper we prove the above theorem in which U need not be bounded. We also replace the above boundary condition by other equivalent conditions to prove the same result.

Now, we introduce our notations and definitions:

Definition 1.1. Let D be a bounded subset of a metric space X. Define the measure of non compactness $\alpha(D)$ of D by

$$\alpha(D) = inf\,\{\epsilon > 0 : D \text{ admits a finite covering of subsets of diameter } \leq \epsilon.\}$$

Definition 1.2. Let $T : X \to X$ be a continuous mapping of a Banach space X. Then T is called a k- set contraction if for all $A \subset X$ with A bounded, $T(A)$ is bounded and $\alpha(TA) \leq k\alpha(A), 0 < k < 1$.

If $\alpha(TA) < \alpha(A)$, for all $\alpha(A) > 0$, then T is called densifying (or condensing).

In what follows some well-known properties of α will be needed, and these are: $\alpha[A] \leq \alpha[B]$ whenever $A \subset B, \alpha[A \cup B] = \max\{\alpha[A], \alpha[B]\}, \alpha[\overline{co}(A)] = \alpha[A], \alpha[\lambda A] = |\lambda|\,\alpha[A], \alpha[\overline{A}] = \alpha[A], \alpha[A + B] \leq \alpha[A] + \alpha[B]$ and $\alpha[A] = 0$ if and only if $\overline{A}$ is compact.

Definition 1.3. Let X and Y be normed linear spaces. A map $F : X \to Y$ is called compact if $F(X)$ is contained in a compact subset of Y.

Definition 1.4. A map $F : X \subseteq E \to E$ where E is a Banach space is said to be completely continuous if $F(Y)$ is relatively compact for all bounded sets $Y \subseteq X$.

Definition 1.5. Let (X, d) be a metric space with $C \subseteq X$. A mapping $F : C \to X$ is nonexpansive if F satisfies $d(F(x), F(y)) \leq d(x, y)$ for all $x, y \in X$. we remark that a nonexpansive map is also a 1-set contraction.

Definition 1.6. A map $f : X \to X$ where$X$ is a Banach space, is said to be demiclosed if for every sequence $x_n \in X$ which converges weakly to x in X(denote by $x_n \rightharpoonup x$) and $f(x_n)$ converges strongly to y, we have $y = f(x)$.

The definitions stated above can be found in [1, 2 or 3].

2 Fixed point theorem for condensing map

Theorem 2.1. *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $p \in U$. Suppose $F : \overline{U} \to C$ is a continuous condensing map with $F(\overline{U})$ a bounded set in C and $F/_{\partial U} = p$. Then F has a fixed point in $\overline{U}$.*

Proof. Define a map $N : C \to C$ by

$$N(x) := \begin{cases} F(x), & x \in \overline{U}, \\ p, & x \in C \setminus \overline{U}. \end{cases}$$

Now $N : C \to C$ is continuous and $N(C)$ is bounded in C as $F(\overline{U})$ is bounded in C. To show that N is a condensing map, let A be a bounded subset of C with $\alpha(A) > 0$. Then since

$$N(A) \subseteq F(\overline{U} \cap A) \cup \{p\}$$

we have that

$$\alpha(N(A) \leq \alpha F(\overline{U} \cap A) \leq \alpha(F(A)) < \alpha(A).$$

Therefore $N : C \to C$ is a condensing map. Hence by the fixed point theorem of Sadovskii [1, Remark 4.3, p. 40] there exists an $x \in C$ with $x = N(x)$. In fact $x \in U$ since $p \in U$ and therefore $x = N(x) = F(x)$. □

3 Fixed point results for compact maps and contractive maps

Theorem 3.1. *Let E be a Banach space, C a convex subset of E, U an open subset of C and $p \in U$. Suppose that $F : \overline{U} \to C$ is a continuous compact (that is, $F(\overline{U})$ is relatively compact subset of C) map. Then either*
$(A_1)F$ has a fixed point in $\overline{U}$ or
(A_2) there is a $u \in U$(the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)p$.

Proof. Suppose (A_2) does not hold and F has no fixed point on ∂U (otherwise we are finished). Then $u \neq \lambda F(u) + (1-\lambda)p$ for $u \in \partial U$ and $\lambda \in [0,1]$. Consider

$$A := \left\{x \in \overline{U} : x = tF(x) + (1-t)p \text{ for some } t \in [0,1]\right\}.$$

Since $p \in U, A \neq \emptyset$. Also the continuity of F implies that A is closed. And $A \cap \partial U = \emptyset$, hence by Urysohn's lemma there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(A) = 1$ and $\mu(\partial U) = 0$. Let

$$N(x) := \begin{cases} \mu(x)F(x) + (1-\mu(x))p, & x \in \overline{U}, \\ p, & x \in C \setminus \overline{U}. \end{cases}$$

Now it is clear that $N : C \to C$ is a continuous map. Then since $N(C) \subseteq \overline{co}(F(\overline{U}) \cup \{p\}$ and the closed convex hull of a compact set in a Banach space is compact, we have that $N : C \to C$ is a continuous compact map. Applying Schauder's fixed point theorem [2, Theorem 3.2, p. 119] we get a $y \in C$ with $y = N(y)$. But $N(y) = p$ if $y \in C \setminus \overline{U}$ and so $y \in \overline{U}$. Thus

$$y = N(y) = \mu(y)F(y) + (1 - \mu(y))p.$$

It follows that $y \in A$. Since $\mu(A) = 1$ we get $\mu(y) = 1$. This implies that $y = F(y)$. □

Remark 3.2. An alternate proof of the above theorem is given in [2,Theorem 5.2, p. 123]

Corollary 3.3 (1.Theorem 5.1, p.48). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $p \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*
$(A_1)F$ has a fixed point in $\overline{U}$, or
(A_2) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1 - \lambda)p$.

Theorem 3.4. *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $p \in U$. Suppose that $F : \overline{U} \to C$ is given by $F := F_1 + F_2$ and $F(\overline{U})$ is a bounded set in C. In addition assume that $F_1 : \overline{U} \to C$ is completely continuous compact mapping and $F_2 : \overline{U} \to C$ is a continuous, k-set $(0 \leq k < 1)$ contractive map.Then either*
$(A_1)F$ has a fixed point in $\overline{U}$ or
(A_2) there exist $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1 - \lambda)p$.

Proof. Let A be a bounded subset of $\overline{U}$ with $\alpha(A) > 0$. Then

$$\alpha(F(A)) = \alpha((F_1 + F_2)A)$$

$$= \alpha(F_1(A) + F_2(A)) \leq \alpha(F_1(A)) + \alpha(F_2(A)).$$

Since F_1 maps bounded sets to pre-compact sets we have $\alpha(F_1(A)) = 0$. Therefore we get,

$$\alpha F(A) \leq \alpha F_2(A) \leq k\alpha(A),$$

which implies that F is a k -set contractive map. Hence by [1, Theorem 5.7, p.56] we have either (A_1) or (A_2). Hence the result. □

Theorem 3.5. *Let X be a Banach space, U an open subset of X with $0 \in U$ and $F : \overline{U} \to X$ a contractive map with $F(\overline{U})$ bounded. Assume that for every $x \in \partial U$, any one of the following conditions is satisfied.*
i) $||F(x)|| \leq ||x||$,
ii)$||F(x)|| \leq ||x - F(x)||$,
iii)$||F(x)|| \leq \{||x||^2 + ||x - F(x)||^2\}^{1/2}$
iv)$||F(x)|| \leq max\{||x||, ||x - F(x)||\}$.
Then F has a unique fixed point.

Proof. We prove the result with (iv) holding. If F does not have a fixed point, then by [1, Theorem 3.2, p. 21] there exist $\lambda \in (0, 1)$ and $x \in \partial U$ with $x = \lambda F(x)$. In particular $F(x) \neq 0$ and by condition (iv) we would have

$$||F(x)|| \leq max\{||\lambda F(x)||, ||\lambda F(x) - F(x)||\}$$

$$= max\{||\lambda F(x)||, (1 - \lambda)||F(x)||\}$$

which gives a contradiction as $\lambda \in (0, 1)$. Therefore F has a fixed point. For uniqueness of the fixed point, let us assume that $F(x) = x, F(y) = y$ and $x \neq y$. Thus

$$d(x, y) = d(F(x), F(y)) < d(x, y),$$

a contradiction. Therefore $x = y$ and hence F has a unique fixed point. With similar arguments as in (iv) we can prove the theorem for other conditions too. □

Remark 3.6. A similar result is proved in [2, Corollary 1.4.2, p.14] where the range of the contractive map F is taken to be a convex subset of X.

4 Fixed points for nonexpansive maps

Theorem 4.1. *Let $\overline{U}$ be a closed convex subset of a uniformly convex Banach space X with $0 \in U$ and $F : \overline{U} \to X$ is a nonexpansive map with $F(\overline{U})$ bounded. Then either*
A_1)*F has a fixed point in $\overline{U}$ or*
A_2) *there exist $\lambda \in (0, 1)$ and $u \in \partial U$ with $u = \lambda F(u)$*
is true.

Proof. Suppose (A_2) does not hold. For $n \geq 2$ define

$$F_n := (1 - \frac{1}{n})F : \overline{U} \to X$$

Then F_n is a contraction map with contraction constant $(1 - \frac{1}{n})$. Since $F(\overline{U})$ is bounded $F_n(\overline{U})$ is bounded. Therefore by [1, Theorem, 3.2, p. 21], either F_n has a fixed point in $\overline{U}$ or there exist $\lambda \in (0, 1)$ and $u \in \partial U$ with $u = \lambda F_n(u)$. Suppose the latter is true, then

$$u = \lambda(1 - \frac{1}{n})F(u) = \lambda_1 F(u), \text{ where } \lambda_1 \in (0, 1) \text{ and } u \in \partial U$$

which is a contradiction since (A_2) does not hold. Consequently for each $n \in \{2, 3..............\}$ we have that F_n has a fixed point $u_n \in U$, that is

$$F_n(u_n) = u_n.$$

Since $F(\overline{U})$ is bounded $F_n(\overline{U})$ is bounded and so the sequence $\{F_n(u_n)\}$ is bounded. Thus we get a bounded sequence $\{u_n\}$ in X. Since X is reflexive, every bounded sequence of elements of X contains a weakly convergent subsequence. Also we know that, a closed convex subset of a Banach space is weakly closed. Therefore there exists a subsequence S of integers and $u \in \overline{U}$ with $u_n \rightharpoonup u$ as $n \to \infty$ in S. In addition since $u_n = (1 - \frac{1}{n})F(u_n)$ we have $||(I - F)(u_n)|| = \frac{1}{n}||F(u_n)|| \to 0$ as $n \to \infty$. Thus $(I - F)(u_n)$ converges strongly to $'0'$. The demiclosedness of $I - F$ implies that $(I - F)u = 0$, which implies that $u = F(u)$. □

Remark 4.2. In the above theorem we have relaxed boundedness on U given in [1, Theorem 3.3, p. 21].

Theorem 4.3. *Let U be a bounded, open, convex subset of a uniformly convex Banach space X with $0 \in U$, and $F : \overline{U} \to X$ a nonexpansive map. Suppose for all $x \in U$, one of the following conditions holds.*
$i)||F(x)|| \leq ||x||$
$ii)||F(x)|| \leq ||x - F(x)||$
$iii)||F(x)||^2 \leq ||x||^2 + ||x - F(x)||^2$
$iv) \langle x, F(x)\rangle \leq ||x||^2$, where $\langle , \rangle$ is a scalar product in X.
Then F has a fixed point in $\overline{U}$.

Proof. If F has no fixed points, then by [1, Theorem 3.3, p.21] there exist $\lambda \in (0, 1)$ and $x \in \partial U$ with $x = \lambda F(x)$. In particular $F(x) \neq 0$ and hence by (i) we have $||F(x)|| \leq ||\lambda F(x)||$, therefore $1 \leq \lambda$. This is a contradiction, hence F has a fixed point in $\overline{U}$.

Because of (ii) we would have

$$||F(x)|| \leq ||\lambda F(x) - F(x)|| = (1 - \lambda)||F(x)||$$

therefore $1 \leq 1 - \lambda$, which is a contradiction. Similarly by (iii) we get

$$||F(x)||^2 \leq ||\lambda F(x)||^2 + ||\lambda F(x) - F(x)||^2 = (\lambda^2 + (1-\lambda)^2)||F(x)||^2$$

that is, $1 \leq \lambda^2 + (1-\lambda)^2$. This is a contradiction, because $\lambda^2 + (1-\lambda)^2 < \lambda + 1 - \lambda = 1$ for every $0 < \lambda < 1$. Condition (iv) implies that $\langle F(x), \lambda F(x)\rangle \leq ||\lambda F(x)||^2$ that is,

$$\lambda \langle F(x), F(x)\rangle \leq \lambda^2 ||F(x)||^2 = \lambda^2 \langle F(x), F(x)\rangle$$

therefore $1 \leq \lambda$, which is a contradiction. Hence F has a fixed point in $\overline{U}$. □

Corollary 4.4. $[1, Theorem 2.6, p.17]$. *Let H be a real Hilbert space, $\overline{B_r} = \{x \in H : ||x|| \leq r\}$ with $r > 0$, and let $F : \overline{B_r} \to H$ be nonexpansive. Suppose for all $x \in \partial\overline{B_r}$ one of the following conditions holds.*
$i)||F(x)|| \leq ||x||$,
$ii)||F(x)|| \leq ||x - F(x)||$,
$iii)||F(x)||^2 \leq ||x||^2 + ||x - F(x)||^2$,
$iv)\langle x, F(x)\rangle \leq ||x||^2$,
Then F has a fixed point in $\overline{B_r}$.

References

[1] R.P. Agarwal, Maria Meehan and D.O' Regan, Fixed point theory and applications, Cambridge University Press, 2001.

[2] A. Granas and J. Dugundji, Fixed point theory, Springer Monographs in Mathematics, 2003.

[3] Sankatha Singh, Bruce Watson and Pramila Srivastava, Fixed point theory and best approximation: The KKM-Map principle, Kluwer Academic Publishers, 1997.

Received: July 10, 2007