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Fixed Point Theorems for Multifunctions on Nonconvex Sets

Neeta Singh

Department of Mathematics
University of Allahabad
Allahabad-211002, India
n_s32132@yahoo.com

Abstract

We provide generalizations of results by K.Yanagi in the sense that the domain of our multivalued maps need not be convex or starshaped.

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1 Introduction and Preliminaries

In recent years many interesting results have been obtained for multivalued mappings. In most of these results the domain of the multivalued map is a convex set. In this paper, our aim is to remove the 'convex' or 'starshaped' conditions imposed on the domain of the functions from well known theorems.

We introduce some necessary notations and definitions.

Throughout the paper, X will denote a Banach space, $CB(X)$ denotes the family of nonempty bounded closed subsets of X and $C(X)$ denotes the family of nonempty compact subsets of X and ∂K will stand for the boundary of K. We write $x_n \rightharpoonup x$ to indicate that the sequence x_n converges weakly to x and $x_n \to x$ will symbolize as usual strong convergence.

Definition 1.1. Let D be the Hausdorff metric on $CB(X)$ induced by the norm of X and let $K \in CB(X)$. $T : K \to CB(X)$ is said to be *nonexpansive* if $D(T(x), T(y)) \leq \|x - y\|$ for every $x, y \in K$. $T : K \to CB(X)$ is said to be a *contraction* if for every $x, y \in K, D(T(x), T(y)) \leq k \|x - y\|$, where $0 \leq k < 1$. $T : K \to CB(X)$ is said to be a *generalized contraction* if for each $x \in K$ there is a number $\alpha(x) < 1$ such that $D(Tx), T(y)) \leq \alpha(x) \|x - y\|$ for every $y \in K$.

Definition 1.2. Let K be a nonempty subset of X. For $x \in K$ we define the inward set of x relative to K, denoted by $I_K(x)$ as follows

$$I_K(x) = \{x + \alpha(y - x) \mid y \in K, \alpha \geq 1\}.$$

We say that a mapping $f : K \to X$ is *weakly inward* if $f(x)$ belongs to the closure of $I_K(x)$ for each $x \in K$.

Definition 1.3. For a nonempty subset K of X and a bounded sequence $\{x_n\}$ in X we define

$$AR(K, \{x_n\}) = \inf\left\{\limsup_{n\to\infty} \|y - x_n\| : y \in K\right\}$$

and

$$A(K, \{x_n\}) = \left\{z \in K \mid \limsup_{n\to\infty} \|z - x_n\| = AR(K, \{x_n\})\right\}$$

The set $A(K, \{x_n\})$ and the number $AR(K, \{x_n\})$ are called respectively, the *asymptotic center* and the *asymptotic radius* of $\{x_n\}$ relative to K.

Definition 1.4. Let K be a nonempty subset of X. $T : K \to CB(X)$ is said to be *demiclosed* on K if $x_n \rightharpoonup x, y_n \to y$ and $y_n \in T(x_n)$ imply $y \in T(x)$.

In section 2 we prove fixed point theorems for nonexpansive multivalued maps and for multivalued contractions.

2 Fixed point theorems for multivalued non-expansive and contraction maps

Theorem 2.1. *Let K be a nonempty weakly compact subset of a Banach space X. Let $T : K \to C(X)$ be nonexpansive and weakly inward. If $I - T$ is demiclosed on K, then T has a fixed point.*

Proof. Without loss of generality, we assume that $0 \in K$. We choose a sequence $\{k_n\}, 0 < k_n < 1$ such that $k_n \to 1 (n \to \infty)$. We define $T_n : K \to C(X)$ by $T_n = k_n T$, for each n and each $x \in K$.

Since for each n and any $x, y \in K$, we have $T_n(x) \subset \overline{I_K(x)}$ and

$$D(T_n(x), T_n(y)) \leq k_n \|x - y\|$$

by applying [4, Theorem 2.1], there exists $x_n \in K$ such that $x_n = T_n x_n = k_n T x_n$. We now take $y_n \in T(x_n)$ satisfying $x_n = k_n y_n$.

Now K being weakly compact, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in K$. Also

$$\|x_{n_i} - y_{n_i}\| = \left(\frac{1}{k_{n_i}} - 1\right) \|x_{n_i}\| \to 0$$

Since $(I - T)$ is demiclosed, we get $0 \in (I - T)(z)$, that is $z \in T(z)$. □

Corollary 2.2. *[3, Theorem 1]Let K be a nonempty weakly compact convex subset of a Banach space X and let $T : K \to C(X)$ be nonexpansive such that $T(x) \subset cl(I_K(x))$ for each $x \in K$. If $I - T$ is demiclosed on K, then T has a fixed point.*

Theorem 2.3. *If K is a nonempty weakly compact subset of a Banach space X and $T : K \to C(K)$ is a nonexpansive mapping such that $I - T$ is demiclosed, then T has a fixed point in K.*

Proof. Without loss of generality we assume that $0 \in K$. We choose a sequence $\{k_n\}, 0 < k_n < 1, k_n \to 1$. We define $T_n : K \to C(K)$ by $T_n = k_n T$, then T_n is a contraction and by [2, Theorem 5], there exists $x_n \in K$ such that $x_n \in T_n(x_n)$.

Since K is weakly compact, there is a subsequence of $\{x_n\}$, again denoted by $\{x_n\}$ converging weakly to $x \in K$. We can write $x_n = k_n z_n$ where $z_n \in T(x_n)$. Hence $\|x_n - z_n\| = (1 - k_n) \|z_n\|$. It follows that

$$y_n = x_n - z_n \in (I - T)(x_n) \to 0 \text{ as } k_n \to 1.$$

Since $I - T$ is demiclosed and $x_n \rightharpoonup x, y_n \to 0$, we get $0 \in (I - T)(x)$, which implies that $x \in T(x)$. □

Theorem 2.4. *Let K be a nonempty weakly compact subset of a uniformly and metrically convex Banach space X and let $T : K \to C(X)$ be nonexpansive. If $T(\partial K) \subset K$ and $\lambda x + (1-\lambda)T(x) \subset K$ for some $\lambda \in (0,1)$, then T has a fixed point.*

Proof. Without loss of generality we assume that $0 \in K$ and choose a sequence $\{k_n\}, 0 < k_n < 1$ such that $k_n \to 1 (n \to \infty)$. By Assad and Kirk[1, Theorem 1] the contraction mapping $T_n : K \to C(X)$ defined by $T_n = k_n T$ has a fixed point x_n. Hence there exists $y_n \in T(x_n)$ satisfying $x_n = k_n y_n$, then

$$\|x_n - y_n\| = \left(\frac{1}{k_n} - 1\right) \|x_n\| \to 0 (n \to \infty)$$

by virtue of the boundedness of K.

Rest of the proof follows exactly as in [3, Theorem 2] by putting $x_0 = 0$ □

Corollary 2.5. *[3, Theorem 2]Let K be a nonempty weakly compact star-shaped subset of a uniformly convex Banach space X and let $T : K \to C(X)$ be nonexpansive. If for each $x \in \partial K, T(x) \subset K$ and $\lambda x + (1-\lambda)T(x) \subset K$ for some $\lambda \in (0,1)$, or $T(x) \subset int(K)$, then T has fixed point.*

Theorem 2.6. *Let K be a nonempty weakly compact subset of a metrically convex Banach space X and $T : K \to C(X)$ be a generalized contraction. If $T(\partial K) \subset K$, then T has a fixed point.*

Proof. As in Theorem 2.4 by [1, Theorem 1] we obtain x_n as a fixed point of T_n. Hence there exists $y_n \in T(x_n)$ satisfying $x_n = k_n y_n$.

Rest of the proof follows exactly as in [3, Theorem 3] by putting $x_0 = 0$. □

Corollary 2.7. *[3, Theorem 3]Let K be a nonempty weakly compact star-shaped subset of a Banach space X and $T : K \to C(X)$ be a generalized contraction. If for each $x \in \partial K, T(x) \subset K$, then T has a fixed point.*

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