Int. Journal of Math. Analysis, Vol. 1, 2007, no. 26, 1249 - 1260

On Subspaces of CAT(κ) Spaces

Assa-Aree Sama-Ae

Department of Mathematics and Computer Science
Faculty of Science and Technology, Prince of Songkla University
Pattani campus, Pattani, Thailand, 94000
sassaare1@hotmail.com

Abstract

In this work, we give a necessary and condition for a complete metric space to be CAT(κ) and then we give a remark on bounded subsets of CAT(κ) spaces and thus we use that fact to show a new proof of the unigueness of circumcenters of bounded sets in complete CAT(κ) spaces. We also give some remarks on the projection onto complete convex subspaces of CAT(κ) spaces.

Mathematics Subject Classification: 51K10, 53C20, 53C22

Keywords: CAT(κ), circumcenter, projection

1 Introduction

Alexandrov introduced the concept of the lower and upper curvature bounds on some metric spaces without the Riemannian structure as generalized Riemannian manifolds [3, 5]; see also [6, 7, 8, 9, 11]. This idea has been very fruitful because it extended many concepts to a much larger class of metric spaces. A Riemannian manifold for which the curvature is bounded above by κ is an example of a space of curvature bounded above by κ. A space of curvature $\leq \kappa$ is not necessarily a Riemannian manifold.

The terminology of CAT(κ) spaces was coined by M. Gromov in 1987. The initials are in honour of E. Cartan, A.D. Alexandrov and V.A. Toponogov. A geodesic metric space X is said to be a CAT(κ) space if for any geodesic triangle of appropriate size is not fatter than its comparison triangle in the model space M_κ^2. A metric space X is said to be of curvature bounded above by κ if it is locally a CAT(κ) space. CAT(0) spaces are generalizations of Hadamard manifolds, which are simply connected, complete Riemannian manifolds such that the sectional curvature is nonpositive. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. The classical hyperbolic spaces, Euclidean buildings (see [10]) and the complex Hilbert ball with a hyperbolic metric (see [15]) are examples of CAT(0) spaces. For more discussion of these spaces and of the fundamental role they play in Geometry, see Bridson and Haefliger [9] and Burago et al. [11].

The following studies, closely related to ours, are worth mentioning: Alexander and Bishop [1] proved comparison and rigidity theorems for curves of bounded geodesic curvature, Maneesawarng and Lenbury [17] studied total curvature and length estimate for curves, Lang and Schroeder [16] studied Jung's theorem, Nigolaev [18] studied the tangent cones, Sama-Ae and Maneesawarng [19] studied geometry of curves on spheres, Chen [13] studied warped products, and the Gauss equation for subspaces was proved by Alexander and Bishop [2]. Chaoha and Phon-on [12] gave a note on fixed point sets in CAT(0) spaces.

In this work, we give a necessary and condition for a complete metric space to be CAT(κ), for a simple case see [7] page 23. Then we give a remark on bounded subsets of CAT(κ) spaces and thus we use that fact to show a new proof of the unigueness of circumcenters of bounded sets in complete CAT(κ) spaces. Actually the uniqueness of circumcenters of bounded sets was proved differently in [9], page 179 and in the paper of Lang and Schroeder in [16]. In the last section, we give some remarks on the projection onto complete convex subspaces of CAT(κ) spaces.

2 Preliminaries

Let (X, d) be a metric space and $\gamma : [a, b] \to X$ a curve in X. The *length* $\ell(\gamma)$ of γ is defined by

$$\ell(\gamma) = \sup \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_k = b$ of $[a, b]$. Then

$$d_i(x, y) := \inf\{\ell(\gamma) | \gamma \text{ is a curve from } x \text{ to } y\}, \text{ for all } x \text{ and } y \text{ in } X,$$

defines a metric on X with distance values in $[0, \infty]$. We call d_i the *inner metric* on X induced by d. A curve $\gamma : [a, b] \to X$ is called *minimizing* or *shortest* if $\ell(\gamma) = d(\gamma(a), \gamma(b))$. It is said to be a *geodesic* if it, in addition, has constant speed. We shall denote by $[x, y]$ a specifically chosen geodesic from x to y.

For convenience, throughout this work we let $\lambda = \sqrt{|\kappa|}$ if $\kappa \neq 0$. For $\kappa \in \mathbf{R}$ let M_κ^2 denote the complete, connected, simply connected, Riemannian

2-manifold of constant Gaussion curvature κ which we shall call a model space and let

$$D_\kappa := \mathrm{diam} M_\kappa^2 = \begin{cases} \frac{\pi}{\lambda} & \text{for } \kappa > 0, \\ \infty & \text{for } \kappa \leq 0. \end{cases}$$

A triangle $\triangle(p,q,r)$ in X is a triangle with points p,q,r as its vertices and three chosen geodesics $[p,q]$, $[q,r]$, $[p,r]$ as its sides. A *comparison triangle* in M_κ^2 for a geodesic triangle $\triangle(p,q,r)$ in X is a triangle $\overline{\triangle} = \triangle(p',q',r')$ in M_κ^2 such that $d(p,q) = d'(p',q')$, $d(q,r) = d'(q',r')$ and $d(p,r) = d'(p',r')$. Such a triangle $\overline{\triangle} \subset M_\kappa^2$ always exists if $d(p,q) + d(q,r) + d(r,p) < 2D_\kappa$ and it is unique up to isometries. Given a pair of a triangle $\triangle(p,q,r)$ in X and its comparison triangle $\triangle(p',q',r')$ in M_κ^2, the *comparison point* for a point $x \in [q,r]$ is the point denoted by x' in $[q',r']$ such that $d(q,x) = d'(q',x')$. A triangle $\triangle$ of perimeter less than $2D_\kappa$ in X is said to satisfy the *CAT(κ) inequality* if for each pair $x,y \in \triangle$, $d(x,y) \leq d'(x',y')$. X is called a *CAT(κ) space* if for all pairs $x,y \in X$ with $d(x,y) < D_\kappa$ there exists a geodesic from x to y and if all triangles in X satisfy the CAT(κ) inequality. A complete CAT(0) space is called a *Hadamard space.* A metric space X is said to be of *curvature* $\leq \kappa$ if it is locally a CAT(κ) space.

The angle at a common endpoint of two geodesics in a CAT(κ) space can be defined as follows. Let $c_1 : [0,a_1] \to X$ and $c_2 : [0,a_2] \to X$ be two geodesics in X with $c_1(0) = c_2(0) = p$. Given $t_1 \in (0,a_1]$ and $t_2 \in (0,a_2]$, we consider the interior angle, denoted by $\overline{\angle}_p(c_1(t_1), c_2(t_2))$, at the point corresponding to p of a comparison triangle of $\triangle(p, c_1(t_1), c_2(t_2))$. The *angle* between geodesics c_1 and c_2 is the number in $[0,\pi]$ defined by:

$$\angle(c_1,c_2) := \lim_{t_1,t_2\to 0} \overline{\angle}_p(c_1(t_1), c_2(t_2)) = \lim_{t\to 0} \overline{\angle}_p(c_1(t), c_2(t)).$$

If $p \neq q$ and $p \neq r$, the *angle* $\angle_p(q,r)$ at p of $\triangle(p,q,r)$ can be defined to be the angle between the geodesics $[p,q]$ and $[p,r]$.

If a, b and c are arc lengths of the sides of a geodesic triangle in M_κ^2 with opposite angles α, β and γ, respectively, then the following are the laws of cosines,

$$\begin{array}{rcll} \cosh(a\lambda) & = & \cosh(b\lambda)\cosh(c\lambda) - \sinh(b\lambda)\sinh(c\lambda)\cos\alpha, & \text{if} \quad \kappa < 0, \\ a^2 & = & b^2 + c^2 - 2bc\cos\alpha, & \text{if} \quad \kappa = 0, \\ \cos(a\lambda) & = & \cos(b\lambda)\cos(c\lambda) + \sin(b\lambda)\sin(c\lambda)\cos\alpha, & \text{if} \quad \kappa > 0. \end{array}$$

Now we state a well-known property of CAT(κ) spaces.

Theorem 2.1 [4, 9] *In a CAT(κ) space, the angle between any two geodesics at their common endpoints exists. If α_1, α_2 and α_3 are angles of a triangle in a CAT(κ) space corresponding respectively to angles α_1', α_2' and α_3' of its comparison triangle in M_κ^2, then $\alpha_i \leq \alpha_i'$ for $i = 1,2,3$. An equality holds for some i if and only if the two triangles bound totally geodesic surfaces isometric to each other.*

3 Subsets of $CAT(\kappa)$ Spaces

In what follows, we let (X, d) be a metric space and d' the metric on M_κ^2. We firstly give a necessary and condition for a complete metric space to be CAT(κ). Actually it is a generalization of the fact proved in [7].

Theorem 3.1 *If X is a CAT(κ) space, then for any pair x, y of points in X, there is a midpoint m between x and y such that*

$$d(m,z) \leq \begin{cases} \frac{1}{\lambda}\cosh^{-1}\left(\frac{\cosh(\lambda d(x,z)) + \cosh(\lambda d(y,z))}{2\cosh\frac{\lambda d(x,y)}{2}}\right) & \text{if } \kappa < 0 \\ \sqrt{\frac{1}{2}d^2(x,z) + \frac{1}{2}d^2(y,z) - \frac{1}{4}d^2(x,y)} & \text{if } \kappa = 0 \\ \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos(\lambda d(x,z)) + \cos(\lambda d(y,z))}{2\cos\frac{\lambda d(x,y)}{2}}\right) & \text{if } \kappa > 0, \end{cases} \quad (1)$$

for all $z \in X$ $(d(x,y) + d(y,z) + d(z,x) < 2D_\kappa$ if $\kappa > 0)$.

And necessity is true if X is a complete metric space.

Proof. Suppose that X is a CAT(κ) space. Let x, y and z be points in X and m a midpoint between x and y. Now we prove in the case $\kappa > 0$. On M_κ^2, we let $\triangle(x', y', z')$ be a comparison triangle of $\triangle(x, y, z)$, m' a comparison point of m and $\alpha = \angle_{m'}(x', z')$. By the law of cosines in M_κ^2, we have that

$$\begin{aligned} \cos(\lambda d'(x',z')) = &\cos(\lambda d'(x',m'))\cos(\lambda d'(m',z')) \\ &+ \sin(\lambda d'(x',m'))\sin(\lambda d'(m',z'))\cos\alpha, \end{aligned} \quad (2)$$

and

$$\begin{aligned} \cos(\lambda d'(y',z')) = &\cos(\lambda d'(x',m'))\cos(\lambda d'(m',z')) \\ &+ \sin(\lambda d'(x',m'))\sin(\lambda d'(m',z'))\cos(\pi - \alpha). \end{aligned} \quad (3)$$

Combining Equations (2) and (3), we get

$$\cos(\lambda d'(x',z')) + \cos(\lambda d'(y',z')) = 2\cos(\lambda d'(x',m'))\cos(\lambda d'(m',z')),$$

which implies

$$d'(m',z') = \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos(\lambda d'(x',z')) + \cos(\lambda d'(y',z'))}{2\cos(\lambda d'(x',m'))}\right). \quad (4)$$

As $d(m,z) \leq d'(m',z')$, $d(x,y) = d'(x',y')$, $d(x,z) = d'(x',z')$, $d(z,y) = d'(z',y')$, and m' is the midpoint of x' and y', Equation (4) becomes

$$d(m,z) \leq \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos(\lambda d(x,z)) + \cos(\lambda d(y,z))}{2\cos\frac{\lambda d(x,y)}{2}}\right),$$

as required. For the case $\kappa \le 0$, we use the law of cosines for triangles in the model spaces involved and prove in the same way as the first case. Hence sufficiency is completely proved.

Next we shall prove the necessity. Let X be a complete metric space and we assume that for any pair x, y of points in X there is a midpoint $m \in X$ between x and y satisfying inequality (1). We shall show that X is a CAT(κ) space. For the case $\kappa = 0$, see [7] on page 23. Now we let $\kappa > 0$ be fixed. Then, we have that X is a D_κ-geodesic, i.e., for any pair x and y with $d(x, y) < D_\kappa$ there is a geodesic segment joining x to y, see [9] on page 4. We now show that X satisfies the CAT(κ) inequality. Let $\triangle(u, v, w)$ be a triangle in X with $d(u, v) + d(v, w) + d(w, u) < 2D_\kappa$. Let t be a midpoint between u and v, $\triangle(u', v', w')$ a comparison triangle of $\triangle(u, v, w)$ and t' a midpoint between u' and v'. By inequality (1), we have

$$d(t,w) \le \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos(\lambda d(u,w)) + \cos(\lambda d(v,w))}{2\cos\frac{\lambda d(u,v)}{2}}\right). \tag{5}$$

As $d(u, v) = d'(u', v')$, $d(v, w) = d'(v', w')$ and $d(u, w) = d'(u', w')$, inequality (5) becomes

$$d(t,w) \quad \le \quad \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos(\lambda d'(u',w')) + \cos(\lambda d'(u',w'))}{2\cos\frac{\lambda d'(u',v')}{2}}\right) = d'(t',w').$$

Therefore X is a CAT(κ) space. With the same method, we can prove the result for the case $\kappa < 0$. This completes the proof.

For a bounded subset A of X, we let $r(A) = \inf_{x\in X} r_x(A)$, where $r_x(A) := \sup_{y\in A} d(x, y)$.

Theorem 3.2 *Let X be a CAT(κ) space and A a bounded subset of X. Then for any pair x, y of points in X satisfies the following inequality:*

$$d(x,y) \le \begin{cases} \dfrac{2}{\lambda}\cosh^{-1}\left(\dfrac{\cosh(\lambda r_x(A)) + \cosh(\lambda r_y(A))}{2\cosh(\lambda r(A))}\right) & \text{if} \quad \kappa < 0 \\ \sqrt{2[r_x(A)]^2 + 2[r_y(A)]^2 - 4[r(A)]^2} & \text{if} \quad \kappa = 0 \\ \dfrac{2}{\lambda}\cos^{-1}\left(\dfrac{\cos(\lambda r_x(A)) + \cos(\lambda r_y(A))}{2\cos(\lambda r(A))}\right) & \text{if} \quad \kappa > 0, \end{cases} \tag{6}$$

we assume furthermore that $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ if $\kappa > 0$.

Proof. Let x, y be two points in X and t a midpoint between x and y. By Theorem 3.1, we have that inequality (1) holds for points x, y and t. We now

prove in three possibilities.
Case $\kappa = 0$. We have

$$d(t,z) \leq \sqrt{\frac{1}{2}d^2(x,z) + \frac{1}{2}d^2(y,z) - \frac{1}{4}d^2(x,y)} \text{ for all } z \in A,$$

that is

$$4d^2(t,z) \leq 2d^2(x,z) + 2d^2(y,z) - d^2(x,y) \text{ for all } z \in A.$$

Hence

$$\begin{aligned} \sup_{z\in A} 4d^2(t,z) &\leq \sup_{z\in A} [2d^2(x,z) + 2d^2(y,z) - d^2(x,y)] \\ &\leq 2\sup_{z\in A} d^2(x,z) + 2\sup_{z\in A} d^2(y,z) - d^2(x,y), \end{aligned}$$

and therefore

$$4[r_t(A)]^2 \leq 2[r_x(A)]^2 + 2[r_y(A)]^2 - d^2(x,y),$$

which implies

$$d^2(x,y) \leq 2[r_x(A)]^2 + 2[r_y(A)]^2 - 4[r_t(A)]^2.$$

But we know that $r(A) \leq r_t(A)$, the latest inequality becomes

$$d(x,y) \leq \sqrt{2[r_x(A)]^2 + 2[r_y(A)]^2 - 4[r(A)]^2},$$

as required.
Case $\kappa > 0$. We have

$$d(t,z) \leq \frac{1}{\lambda}\cos^{-1}\left(\frac{\cos(\lambda d(x,z)) + \cos(\lambda d(y,z))}{2\cos\frac{\lambda d(x,y)}{2}}\right) \text{ for all } z \in A.$$

Since the function $\cos x$ decreases on $[0, \frac{\pi}{2}]$, we get that

$$\cos(\lambda d(t,z)) \geq \frac{\cos(\lambda d(x,z)) + \cos(\lambda d(y,z))}{2\cos\frac{\lambda d(x,y)}{2}} \text{ for all } z \in A,$$

which gives

$$\begin{aligned} \cos(\lambda r_t(A)) &= \cos(\lambda \sup_{z\in A} d(t,z)) \\ &= \inf_{z\in A} \cos(\lambda d(t,z)) \\ &\geq \inf_{z\in A}\left(\frac{\cos(\lambda d(x,z)) + \cos(\lambda d(y,z))}{2\cos\frac{\lambda d(x,y)}{2}}\right) \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\inf_{z\in A}\cos(\lambda d(x,z)) + \inf_{z\in A}(\cos\lambda d(y,z))}{2\cos\frac{\lambda d(x,y)}{2}}\\
&= \frac{\cos(\lambda\sup_{z\in A} d(x,z)) + \cos(\lambda\sup_{z\in A} d(y,z))}{2\cos\frac{\lambda d(x,y)}{2}}\\
&= \frac{\cos(\lambda r_x(A)) + \cos(\lambda r_y(A))}{2\cos\frac{\lambda d(x,y)}{2}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\cos\frac{\lambda d(x,y)}{2} &\geq \frac{\cos(\lambda r_x(A)) + \cos(\lambda r_y(A))}{2\cos(\lambda r_t(A))}\\
&\geq \frac{\cos(\lambda r_x(A)) + \cos(\lambda r_y(A))}{2\cos(\lambda r(A))}.
\end{aligned}
$$

Therefore

$$
d(x,y) \leq \frac{2}{\lambda}\cos^{-1}\left(\frac{\cos(\lambda r_x(A)) + \cos(\lambda r_y(A))}{2\cos(\lambda r(A))}\right).
$$

Case $\kappa < 0$. We have

$$
d(t,z) \leq \frac{1}{\lambda}\cosh^{-1}\left(\frac{\cosh(\lambda d(x,z)) + \cosh(\lambda d(y,z))}{2\cosh\frac{\lambda d(x,y)}{2}}\right) \text{ for all } z\in A,
$$

that is

$$
\cosh(\lambda d(t,z)) \leq \frac{\cosh(\lambda d(x,z)) + \cosh(\lambda d(y,z))}{2\cosh\frac{\lambda d(x,y)}{2}} \text{ for all } z\in A.
$$

Then

$$
\begin{aligned}
\sup_{z\in A}\cosh(\lambda d(t,z)) &\leq \sup_{z\in A}\left(\frac{\cosh(\lambda d(x,z)) + \cosh(\lambda d(y,z))}{2\cosh\frac{\lambda d(x,y)}{2}}\right)\\
&\leq \frac{\sup_{z\in A}\cosh(\lambda d(x,z)) + \sup_{z\in A}(\cosh\lambda d(y,z))}{2\cosh\frac{\lambda d(x,y)}{2}}\\
&= \frac{\cosh(\lambda\sup_{z\in A} d(x,z)) + \cosh(\lambda\sup_{z\in A} d(y,z))}{2\cosh\frac{\lambda d(x,y)}{2}}\\
&= \frac{\cosh(\lambda r_x(A)) + \cosh(\lambda r_y(A))}{2\cosh\frac{\lambda d(x,y)}{2}}. \qquad (7)
\end{aligned}
$$

Since

$$
\cosh(\lambda r_t(A)) = \sup_{z\in A}\cosh(\lambda d(t,z)), \qquad (8)
$$

combining (7) and (8), we have

$$\cosh(\lambda r_t(A)) \le \frac{\cosh(\lambda r_x(A)) + \cosh(\lambda r_y(A))}{2\cosh\frac{\lambda d(x,y)}{2}}.$$

Therefore

$$\begin{aligned}\cosh\frac{\lambda d(x,y)}{2} &\le \frac{\cosh(\lambda r_x(A)) + \cosh(\lambda r_y(A))}{2\cosh(\lambda r_t(A))}\\ &\le \frac{\cosh(\lambda r_x(A)) + \cosh(\lambda r_y(A))}{2\cosh(\lambda r(A))},\end{aligned}$$

which gives

$$d(x,y) \le \frac{2}{\lambda}\cosh^{-1}\left(\frac{\cosh(\lambda r_x(A)) + \cosh(\lambda r_y(A))}{2\cosh(\lambda r(A))}\right),$$

as required.

We shall utilize the results of Theorem 3.1 and 3.2 to give a new proof of the uniqueness of circumcenters of bounded subsets in complete CAT(κ) spaces.

Theorem 3.3 *Let X be a complete CAT(κ) space and $A \subset X$ a bounded subset with $r(A) < \frac{D_\kappa}{2}$. Then there is a unique $x \in X$, the circumcenter of A, such that $r_x(A) = r(A)$.*

Proof. By assumption we have that inequality (6) holds. We now prove that a given sequence (x_n) with $r_{x_n}(A) \to r(A)$ is Cauchy by considering three possibilities:

Case $\kappa > 0$. Since $r(A) = \inf_{x\in X} r_x(A) < \frac{D_\kappa}{2}$, there exists a large positive n^* such that $r_{x_n}(A), r_{x_m}(A)$ and $d(x_n, x_m)$ will not exceed $\frac{D_\kappa}{2}$ for all $n, m > n^*$. Because $r_{x_n}(A) \to r(A)$, we get

$$\frac{\cos(\lambda r_{x_n}(A)) + \cos(\lambda r_{x_m}(A))}{2\cos(\lambda r(A))} \to 1, \text{ as n, m} \to \infty.$$

By using inequality (6), we have $\cos\frac{\lambda d(x_n,x_m)}{2} \to 1$, as $m, n \to \infty$ and then $d(x_n, x_m) \to 0$.

Case $\kappa < 0$. By considering the following inequality

$$1 \le \cosh\frac{\lambda d(x_n, x_m)}{2} \le \frac{\cosh(\lambda r_{x_n}(A)) + \cosh(\lambda r_{x_m}(A))}{2\cosh(\lambda r(A))},$$

we have that $\cosh\frac{\lambda d(x_n,x_m)}{2} \to 1$ as $m, n \to \infty$, which implies $d(x_n, x_m) \to 0$.

Case $\kappa = 0$. It is clear that $d(x_n, x_m) \to 0$ as $m, n \to \infty$ by inequality (6).

By showing above we have that the sequence (x_n) with $r_{x_n}(A) \to r(A)$ is Cauchy. Since X is complete, we have that a circumcenter exists and uniqueness of circumcenter follows directly from the fact of inequality (6).

Remark 3.4 If we change the condition of the space X in Theorem 3.3 from being CAT(κ) to being of curvature $\leq \kappa$, the theorem may not be true. For instance, consider a 2-dimensional cylinder. It is a complete space of curvature ≤ 0 but the circumcenter of bounded tube is not unique.

4 Convex Subspaces and Projections

The projection onto a complete, convex subspace of a CAT(0) spaces X is the name given to the map $p : X \to C$ constructed in the following proposition.

Theorem 4.1 [9] *Let X be a CAT(0) space, and let C be a subset which is complete in the induced metric. Then,*

(1) for every $x \in X$, there exists a unique point $p(x) \in C$ such that $d(x, p(x)) = d(x, C) := \inf_{y \in C} d(x, y)$,

(2) if x' belongs to the geodesic segment $[x, p(x)]$, then $p(x) = p(x')$, and

(3) given $x \in X - C$ and $y \in C$, if $y \neq p(x)$, then $\angle_{p(x)}(x, y) \geq \frac{\pi}{2}$.

Actually we can generalize the projection onto convex subsets of CAT(κ) spaces for arbitrary κ, when $\kappa > 0$ we assume furthermore that C is D_κ-geodesic and we replace X by $V = \{x \in X \mid d(x, C) < \frac{D_\kappa}{2}\}$. See more detail in [9].

Now we shall prove the necessity of Proposition 4.1*(3)* for arbitrary κ. We start the proof with the following lemma.

Lemma 4.2 *Let $\triangle$ be a triangle in M_κ^2 with sides of length $a, b, c > 0$ and angle γ at the vertex opposite to the side of length c. If $c < a \leq D_\kappa$, then $\gamma < \frac{\pi}{2}$.*

Proof. In the case $\kappa = 0$, it is obvious. We are going to show in the case $\kappa > 0$. By the law of cosine in M_κ^2, we have

$$\cos(c\lambda) = \cos(a\lambda)\cos(b\lambda) + \sin(a\lambda)\sin(b\lambda)\cos\gamma. \tag{9}$$

Since $c < a \leq D_\kappa$, we get $\cos(a\lambda) < \cos(c\lambda)$ and therefore Equation (9) becomes

$$\cos(a\lambda) < \cos(a\lambda)\cos(b\lambda) + \sin(a\lambda)\sin(b\lambda)\cos\gamma.$$

Consequently,

$$\begin{aligned}\cos\gamma &> \frac{\cos(a\lambda) - \cos(a\lambda)\cos(b\lambda)}{\sin(a\lambda)\sin(b\lambda)} \\ &= \frac{\cos(a\lambda)(1 - \cos(b\lambda))}{\sin(a\lambda)\sin(b\lambda)} > 0,\end{aligned}$$

which implies $\gamma < \frac{\pi}{2}$.

Lastly, for the case $\kappa < 0$, we use the law of cosine in M_κ^2 for $\kappa < 0$ and prove in the same way as the case $\kappa > 0$. The lemma is then completely proved.

Theorem 4.3 *Let X be a CAT(κ) space, and C a convex subset which is complete in the induced metric (when $\kappa > 0$ we assume that C is $D_\kappa-$geodesic and we replace X by $V = \{x \in X \mid d(x,C) < \frac{D_\kappa}{2}\}$). If $x \in X - C$ and $\angle_z(x,y) \geq \frac{\pi}{2}$, for a fixed point $z \in C$ and for all $y \in C$ with $y \neq z$, then z is the projection of x in C.*

Proof. We shall show that $d(x,z) = d(x,C)$. Suppose that there were some $w \in C$ such that $d(x,w) < d(x,z)$. Consider a comparison triangle $\triangle(x',z',w')$ of the triangle $\triangle(x,z,w)$. Since $d(w',x') < d(z',x')$ by Lemma 4.2, we have that $\angle_{z'}(w',x') < \frac{\pi}{2}$. Therefore $\angle_z(w,x) \leq \angle_{z'}(w',x') < \frac{\pi}{2}$, which is a contradiction. Thereby $d(x,z) = d(x,C)$. This proves the theorem.

Next we generalize the fact in Proposition 3.1 of [14] to CAT(κ) spaces. For a nonempty subset C of X put $I_C(x) = \bigcup_{y \in X} \{y : (x,y] \cap C \neq \phi\} \cup \{x\}$.

Theorem 4.4 *Let C be a nonempty closed convex subset of a CAT(κ) space X, $x \in X$ and $p(x) \in X$ the projection of x in C (when $\kappa > 0$ we assume that C is $D_\kappa-$geodesic and we replace X by $V = \{x \in X \mid d(x,C) < \frac{D_\kappa}{2}\}$). If $y \in \overline{I_C(p(x))} - \{p(x)\}$, then $d(x,p(x)) < d(x,y)$.*

Proof. Let $y \in \overline{I_C(p(x))} - \{p(x)\}$. Then there exists a sequence (y_n) in $I_C(p(x))$ such that $y_n \to y$. For all large n we can find $z_n \in (p(x), y_n] \cap C$. Because $z_n \in C - \{p(x)\}$, $\angle_{p(x)}(x,z_n) \geq \frac{\pi}{2}$. Therefore the angle at $p(x)$ of comparison triangle $\overline{\triangle}(p(x),x,y_n)$ is greater or equal $\frac{\pi}{2}$. By the laws of cosines we have that

$$\begin{array}{rcll} \cosh(\lambda d(x,y_n)) & \geq & \cosh(\lambda d(x,p(x)))\cosh(\lambda d(p(x),y_n)) & \text{if } \kappa < 0, \\ d^2(x,y_n) & \geq & d^2(x,p(x)) + d^2(p(x),y_n) & \text{if } \kappa = 0, \\ \cos(\lambda d(x,y_n)) & \leq & \cos(\lambda d(x,p(x)))\cos(\lambda d(p(x),y_n)) & \text{if } \kappa > 0. \end{array}$$

By taking $n \to \infty$, we have

$$\begin{array}{rcll} \cosh(\lambda d(x,y)) & \geq & \cosh(\lambda d(x,p(x)))\cosh(\lambda d(p(x),y)) & \text{if } \kappa < 0, \\ d^2(x,y) & \geq & d^2(x,p(x)) + d^2(p(x),y) & \text{if } \kappa = 0, \\ \cos(\lambda d(x,y)) & \leq & \cos(\lambda d(x,p(x)))\cos(\lambda d(p(x),y)) & \text{if } \kappa > 0, \end{array}$$

which give

$$\begin{array}{rcll} \cosh(\lambda d(x,y)) & > & \cosh(\lambda d(x,p(x))) & \text{if } \kappa < 0, \\ d^2(x,y) & > & d^2(x,p(x)) & \text{if } \kappa = 0, \\ \cos(\lambda d(x,y)) & < & \cos(\lambda d(x,p(x))) & \text{if } \kappa > 0. \end{array}$$

Therefore we obtain $d(x,p(x)) < d(x,y)$ for any κ, as required.

Acknowledgments

This work is a continuation of part of a Ph.D. thesis guided by Asst. Prof. Dr. Chaiwat Maneesawarng to whom the author is very grateful.

References

[1] S.B. Alexander and R.L. Bishop, Comparison theorems for curves of bounded geodesic curvature in metric spaces of curvature bounded above, Differential Geom. Appl. 9 (1996), 67–86.

[2] S.B. Alexander and R.L. Bishop, Gauss equation and injectivity radii for subspaces in spaces of curvature bounded above, Geometriae Dedicata. 117 (2006), 65-84.

[3] A.D. Alexandrov, A theorem on triangles in a metric space and some of its application, Trudy Mat. Inst. Steklov. 38 (1951), 5–23 (in Russian).

[4] A.D. Alexandrov, Theory of curves based on the approximation by polygonal lines, Nautch. sess. Leningr. Univer., Tesis dokl. na sekth. matem. nauk. (1946).

[5] A.D. Alexandrov, Über eine Verallgemeinerung der Riemannschen Geometrie, Schr. Forsch. Math. 1 (1957), 33–84.

[6] A.D. Alexandrov, V.N. Berestovskii and I.G. Nikolaev, Generalized Riemannian spaces, Russian Math. Survey. 41 (1986), no. 1, 1–54.

[7] W. Ballmann, Lectures on Spaces of Nonpositive Curvature, Birkhäuser, Basel, 1995.

[8] W. Ballman, Singular spaces of Nonpositve curvature, in: E Ghys and P. de la Harpe, eds., Sur les Groups Hyperbolique d́apres Gromov (Birkhäuser, Boston), 1990.

[9] M.R. Bridson, A. Haefliger, Metric spaces of Nonpositive Curvature, Springer, Heidelburg, 1999.

[10] K.S. Brown, Buildings, Springer-Verlag, New York, 1989.

[11] D. Burago, Yu. Burago, S. Ivanov, A Course in Metric Geometry, in: Graduate Stud. Math. 33, Amer. Math. Society, Providence, RI, 2001.

[12] P. Chaoha and A. Phon-on, A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl. 320 (2006), 983–987.

[13] C.H. Chen, Warped products of metric spaces of curvature bounded from above, Trans. Amer. Math. Soc. 351 (1999), 4727–4740.

[14] S. Dhompongsa, A. Keawkhao and B. Panyanak, Lim's theorems for multivalued mappings in CAT(0) spaces, J. Math. Anal. Appl. 312 (2005), 478–487.

[15] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive mapping, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, 1984.

[16] U. Lang, V. Schroeder, Jung's Theorem for Alexandrov spaces of curvature bounded above, Ann. Global Anal. Geom. 15 (1997), 263–275.

[17] C. Maneesawarng, Y. Lenbury, Total curvature and length estimate for curves in CAT(K) spaces, Differential Geom. Appl. 19 (2003), 211–222.

[18] I. Nikolaev, The tangent cone of an Aleksandrov space of curvature $\geq$ K, Manuscripta Math. 86 (1995), 137–147.

[19] A. Sama-Ae and C. Maneesawarng, Geometry of Curves on Spheres in CAT(κ) Spaces, to appear.

Received: July 10, 2007