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Nonoscillatory Properties of Nonlinear Differential Equations of Third Order

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Abstract. The superlinear differential equation $x'''(t) + q(t)f(x(t)) = 0$. $t \geq T_0$ is considered.$q(t) \in C([T_0, \infty))$,without any restriction on its sign and $f \in C(\mathbb{R})$, $y.f(y) > 0$ for $y \neq 0$ and $\int_T^{\infty}[1/f(y)]dy < \infty$ and f has a continuous derivative on $\mathbb{R} - \{0\}$;with $f\prime(y) > 0$ and $f"(y) \geq 0$.for all $y \neq 0$

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1. Introduction

Consider the third order nonlinear ordinary differential equation

$$x'''(t) + q(t)f[x(t)] = 0 \tag{E}$$

where q is continuous real-valued functions on an interval $[T_0\ ,\infty[\ T_0 \geq 0$; and f is a continuous function on the real line $\mathbb{R}$; which is continuously differentiable ,except possibly at 0, and satisfies

$$yf(y) > 0, \ f'(y) \geq 0 \ \text{ for all } \ y \in \mathbb{R} - \{0\}$$

The oscillatory behavior of solutions of third order ordinary differential equations has been the subject of intensive research. Among numerous papers dealing with this subject we refer in particular to Hannan [1]; Lazer [2], Erbe [3 ,4], G.D.Jones [6] for linear equation, and P.Waltman [8], T.Kura [15,16]; Heidel [5] ATiryaki.O.Celebi [13,14], Singh.Y.P [9], Soltes.P, A[10], Ohme [17] for nonlineear equation where $q(t)$ is continuous and does not change its sign on the infinite half-axis $[T_0, \infty[$.

We are interested in the case where no assumption on the sign of the coefficient $q(t)$ is made and (E) is strongly superlinear in the sense that

$$\int^{\infty} \frac{dy}{f(y)} < \infty \text{ and} \int^{-\infty} \frac{dy}{f(y)} < \infty$$

Throughout the paper, we shall restrict our attention only to the solutions of the differentiable equation(E) which exist on some half interval $[T_0, \infty[$.

A solution of (E) will be said to be oscillatory if it changes its sign for arbitrary large values of t, and otherwise it is said to be non-oscillatory. Equation (E) is called oscillatory if all its solutions are oscillatory.

2. Main Results

we denote by S the set of all solutions of (E),
$\Pi = \{x(t) \in S \ ; x(t) \text{ is nonoscillatory}\}$ and R is a positive continuously function on the $[T_0, \infty)$ such that R' is nonnegative and decreasing on $[T_0, \infty)$.

In this paper we shall assume that the following conditions holds.

$$\int^{\infty} \frac{dy}{f(y)} < \infty \text{ and} \int^{-\infty} \frac{dy}{f(y)} < \infty$$

$$\liminf_{t\to\infty} \int_{T_0}^{t} R(s)q(s)ds > -\infty \tag{A1}$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_{T_0}^{t} \int_{T_0}^{s} R(u)q(u)duds = +\infty \tag{A2}$$

$$f((y)f"(y) - 2 \ \ f'^2(y) \leq 0 \quad \text{ for every y} \geq 0 \tag{A3}$$

Remark 1. *The condition A3 holds for* $f(y) \ = y^n, n \geq 0$ *for instance.*

Theorem 1. *Suppose that* $A1, A2, A3$ *holds if* $x(t)$ *is a nonoscillatory solution of* (E) *Then* $\{x(t) \in \Pi \ ; x'(t)x"(t) > 0\} = \varnothing$.

Proof. Let $x(t)$ be a nonoscillatory solution on a interval $[T_0, \infty[, t \geq T_0$ of the differential equation (E). Without loss of generality, we may assume that $x(t) \neq 0$ for all $t \geq T_0$. Furthermore, we suppose that $x(t)$ is positive on $[T, \infty[$ for some $T \geq T_0$.

If $x'(t) < 0$ and $x"(t) < 0$, then this implies that $x(t)$ is negative for large t and whence the set is empty.

If $x'(t) > 0$ and $x"(t) > 0$,then for every $t \geq T$ we define $w(t)$ by :

$$W(t) = R(t)\frac{x"}{f[x(t)]} \qquad t \geq T$$

By differentiation, we obtain for every $t \geq T$,

$$w'(t) = R'(t)\frac{x"(t)}{f[x(t)]} + R(t)\left[-q(t) - \frac{x'(t)x"(t)f'[x(t)]}{f^2[x(t)]}\right]$$

So, for any $t \geq T$

$$\int_T^t R(s)q(s)ds$$

$$= -w(t) + K_0 + \int_T^t R'(s)\frac{x"(s)}{f[x(s)]}ds - \int_T^t R(s)\frac{x'(s)x"(s)f'[x(s)]}{f^2[x(s)]}ds \qquad \text{(A4)}$$

and hence by integrating by part over $[T,t]$,we obtain for $t \geq T$

$$\int_T^t \frac{x'(s)x"(s)f'[x(s)]}{f^2[x(s)]}ds$$

$$= \frac{1}{2}(\frac{x'^2(t)f'[x(t)]}{f^2[x(t)]} - \frac{x'^2(T)f'[x(T)]}{f^2[x(T)]}) + \frac{1}{2}\int_T^t \frac{x'^3(s)}{f[x(s)]}H(s)ds \qquad \text{(A5)}$$

where

$$H(s) = \frac{2f'^2[x(s)] - f[x(s)]f"[x(s)]}{f^2[x(s)]}. > 0.$$

Thus

$$-\int_T^t R(s)\frac{x'(s).x"(s).f.'[x(s)]}{f.^2[x(s)]}.ds = \frac{1}{2}V(t) - D(t) < 0$$

where

$$\begin{aligned} D(t) &= \frac{1}{2}R(t)\frac{x'^2(t)f'[x(t)]}{f^2[x(t)]} - C_0 + \frac{1}{2}\int_T^t \frac{x'^3(s)}{f[x(s)]}H(s)R(s)ds \\ V(t) &= \int_T^t R'(s)\frac{x'^2(s)f'[x(s)]}{f^2[x(s)]}ds > 0 \end{aligned}$$

and

$$C_0 = \frac{1}{2}R(T)\frac{x'^2(T)f'[x(T)]}{f^2[x(T)]}$$

From A4 we have

$$\int_T^t R(s)q(s)ds = -w(t) + K_0 + \int_T^t R'(s)\frac{x"(s)}{f[x(s)]}ds + \frac{1}{2}V(t) - D(t) \qquad \text{(A7)}$$

But

$$\int_T^t R'(s)\frac{x"(s)}{f[x(s)]}ds = \quad R'(t)\frac{x'(t)}{f[x(t)]} - C - \int_T^t R"(s) \quad \frac{x'(s)}{f[x(s)]}ds + V(t) \tag{A8}$$

where $C = \ R'(T)\frac{x'(T)}{f[x(T)]}$
since $\int_{x(T)}^{\infty} \frac{dy}{f(y)} < \infty$, and $R'(t)$ is positive and decreasing , this ensures

$$g(t) = \quad R'(t)\frac{x'(t)}{f[x(t)]} - C - \int_T^t R"(s) \quad \frac{x'(s)}{f[x(s)]}ds$$

is bounded, then there exists a constants m and M such that

$$m \leq g(t) \leq M \tag{A9}$$

Now we have

$$\int_T^t R(s)q(s)ds \leq - \ R(t)\frac{x"(t)}{f[x(t)]} + K_0 + M + \frac{3}{2}V(t) - D(t) \tag{A10}$$

Case1. The integral

$$\int_T^{\infty} R'(s)\frac{x'^2(s)f'(x(s))}{f^2(x(s))}ds$$

is finite. In this case, there exists a positive constant N so that

$$V(t) = \int_T^t R'(s)\frac{x'^2(s)f'(x(s))}{f^2(x(s))}ds \leq N$$

Since $x'(t).x"(t) > 0$, by A9; we get

$$\int_T^t R(s)q(s)ds \leq C$$

where $C = K_0 + M + \frac{3}{2}N + C_0$
This implies:

$$\frac{1}{t}\int_T^t \left[\int_T^s R(u)q(u)du\right] ds \leq C$$

which contradicts A2.
Case 2 .The intgral

$$\int_T^\infty R'(s)\frac{x'^2(s)f'(x(s))}{f^2(x(s))}ds$$

is ininite.Then there are two possibilities

$$\lim_{t\to\infty} .D(t) \quad \text{is finite} \tag{2.1}$$

or

$$\lim_{t\to\infty} .D(t) = \infty \tag{2.2}$$

Case (2.1), from A6 we have

$$\lim_{t\to\infty}\left\{-\int_T^t R(s)\frac{x'(s)x"(s)f'(x(s))}{f^2(x(s))}ds\right\} = \lim_{t\to\infty}\left\{\frac{1}{2}V(t) - D(t)\right\} = +\infty$$

which contradicts $-\int_T^t R(s)\frac{x'(s)x"(s)f'(x(s))}{f^2(x(s))}ds < 0$

Case (2.2). By condition A1 it follows that for some constant λ

$$\lambda \le -\ R(t)\frac{x"(t)}{f[x(t)]} + K_0 + M + \frac{3}{2}V(t) - D(t)$$

From (2.2) there exists a positive constant A so that

$$-\lambda + K_0 + M - D(t) < -A \qquad \text{for every} \quad t \ge T_1 > T$$

The last inequality gives

$$R(t)\frac{x"(t)}{f[x(t)]} \le -A + \frac{3}{2}R'(T)Z(t)$$

where

$$Z(t) = \int_T^t \frac{x'^2(s)f'(x(s))}{f^2(x(s))}ds$$

Now integrate between T_1 and t, to obtain

$$\int_{T_1}^t R(s)\frac{x"(s)}{f[x(s)]}ds$$

$$= R(t)\frac{x'(t)}{f[x(t)]} - K - \int_{T_1}^t R'(s)\frac{x'(s)}{f[x(s)]}ds + \int_{T_1}^t R(s)\frac{x'^2(s)f'(x(s))}{f^2(x(s))}ds$$

$$\le -A(t - T_1) + \frac{3}{2}R'(T_1)\int_{T_1}^t Z(s)ds$$

this implies

$$R(T_1)Z(t) \le K + \int_{T_1}^{t} R'(s)\frac{x'(s)}{f[x(s)]}ds - A(t-T_1) + \frac{3}{2}R'(T_1)\int_{T_1}^{t} Z(s)ds \tag{A13}$$

By the Bonnet theorem, for a fixed $t \ge T_1$ and some $\xi \in [T_1, t]$ we have

$$\int_{T_1}^{t} R'(s)\frac{x'(s)}{f[x(s)]}ds = R'(T_1)\int_{T_1}^{\xi}\frac{x'(s)}{f[x(s)]}ds = R'(T_1)\int_{x(T_1)}^{x(\xi)}\frac{dy}{f(y)}$$

$$\le R'(T_1)\int_{x(T_1)}^{\infty}\frac{dy}{f(y)} = \delta$$

From A13 we have

$$R(T_1)Z(t) \le K + \delta - A(t-T_1) + \frac{3}{2}R'(T_1)\int_{T_1}^{t} Z(s)ds$$

$$\le \frac{3}{2}R'(T_1)\int_{T_1}^{t} Z(s)ds$$

We may use the Gronwall inquality to obtain $Z(t) = 0,$ .for all $t \ge T_1$.But this clearly contradicts the fact that $V(t)$ is infinite and thus $N_2 = \varnothing$. ∎

Lemma 1. *if $x(t)$ is a nonoscillatory solution of (E) such yhat $x'(t). > 0$ and $x"(t) < 0$ then*

$$\int_{T}^{\infty} R'(s)\frac{x'^2(s)f'[x(s)]}{f^2[x(s)]}ds$$

is finite

Proof. From A8 and A9 we have

$$\int_{T}^{t} R'(s)\frac{x"(s)}{f[x(s)]}ds = \quad g(t) + \int_{T}^{t} R'(s)\frac{x'^2(s)f'[(x(s)]}{f^2[x(s)]}ds \le 0$$

This implies

$$0 \le \int_{T}^{t} R'(s)\frac{x'^2(s)f'[(x(s)]}{f^2[x(s)]}ds \le -g(t) \le -m$$

This proves the lemma. ∎

Theorem 2. *Suppose that $A1, A2, A3$ hold and let R be as in Theorem 1.If $x(t)$ is a nonoscillatory solution of (E) then $\{x(t) \in \ \Pi \ \ ; \quad x'(t) > 0, \quad x"(t) < 0\} = \varnothing$.*

Proof. From A10 we have

$$\int_T^t R(s)q(s)ds \leq -\ R(t)\frac{x"(t)}{f[x(t)]} + K_0 + M + \frac{3}{2}V(t) - D(t)$$

By lemma 1 it follows that

$$\int_T^t R(s)q(s)ds \leq -\ R(t)\frac{x"(t)}{f[x(t)]} + \Delta$$

where $\Delta = K_0 + M - \frac{3}{2}m$

The last inequality give

$$\begin{aligned}\int_T^t \int_T^s R(\xi)q(\xi)d\xi ds \ &\leq\ -\ \int_T^t R(s)\frac{x"(s)}{f[x(s)]} + \Delta(t-T)\\ &\leq\ \int_T^t R'(s)\ \ \frac{x'(s)}{f[x(s)]}ds + \Delta(t-T) + K\end{aligned}$$

By the Bonnet theorem ,for a fixed $t \geq T$ and some $\xi \in [T, t]$,we have

$$\int_T^t R'(s)\ \ \frac{x'(s)}{f[x(s)]}ds = R'(T)\int_{x(T)}^{x(\xi)} \frac{dy}{f(y)} \leq R'(T)\int_{x(T)}^{\infty} \frac{dy}{f(y)} = N < \infty$$

Hence, for every $t \geq T$ we have

$$\limsup \frac{1}{t}\int_T^t \int_T^s R(\xi)q(\xi)d\xi ds \leq \Delta \quad \text{ as } \ t \to \infty$$

This contradicts condition A2 ▮

Theorem 3. *Suppose that* $A1, A2, A3$ *hold. If* $x(t)$ *is a nonoscillatory solution of* (E) *such that* $x'(t) < 0$ *and* $x"(t) > 0$

then

$$x"(t) = O([R(t)]^{-1})$$

Proof. Since $x"(t) > 0, x'(t) < 0$ and from A4, A6 we have

$$\begin{aligned}\int_T^t R(s)q(s) &\leq K_0 + \int_T^t R'(s)\frac{x"(s)}{f[x(s)]}ds - \int_T^t R(s)\frac{x'(s)x"(s)f'[x(s)]}{f^2[x(s)]}ds\\ &\leq \int_T^t R'(s)\frac{x'^2(s)f'[x(s)]}{f^2[x(s)]}ds - \int_T^t R(s)\frac{x'(s)x"(s)f'[x(s)]}{f^2[x(s)]}ds + K \qquad \text{(A11)}\\ &\leq \int_T^t [R'(s)\frac{x'(s)}{f[x(s)]} - R(s)\frac{x"(s)}{f[x(s)]}]\frac{x'(s)f'[x(s)]}{f[x(s)]}ds + K\end{aligned}$$

We distinguish two mutually exclusive cases where

$$\int_T^\infty [R'(s)\frac{x'(s)}{f[x(s)]} - R(s)\frac{x"(s)}{f[x(s)]}]\frac{x'(s)f'[x(s)]}{f[x(s)]}ds$$

is finite or infinite.

Case 1. The integral

$$\int_T^\infty \left[R'(s)\frac{x'(s)}{f[x(s)]} - R(s)\frac{x"(s)}{f[x(s)]}\right]\frac{x'(s)f'[x(s)]}{f[x(s)]}ds$$

is finite. In this case there exists a positive constant L so that

$$\int_T^t \left[R'(s)\frac{x'(s)}{f[x(s)]} - R(s)\frac{x"(s)}{f[x(s)]}\right]\frac{x'(s)f'[x(s)]}{f[x(s)]}ds \leq L \quad \text{for every } t \geq T$$

and , by using A11, for every $t \geq T$ we get

$$\int_T^t R(s)q(s)ds \leq L + K = C_1$$

Therefore, for every $t \geq T$ we have

$$\int_T^t \left[\int_T^s R(u)q(u)du\right] ds \leq C_1(t-T)$$

This contradicts condition A2.

Case 2 . The integral

$$\int_T^\infty \left[R'(s)\frac{x'(s)}{f[x(s)]} - R(s)\frac{x"(s)}{f[x(s)]}\right]\frac{x'(s)f'[x(s)]}{f[x(s)]}ds$$

is infinite. There are three possible limits of $x"(t)$ as t tends to infinity.

i) $\lim_{t\to\infty} x"(t) = +\infty$

ii) $\lim_{t\to\infty} x"(t) = c > 0$

iii) $\lim_{t\to\infty} x"(t) = 0$

i) is impossible since it implies that $x'(t)$ is positive for large t.

ii).also implies that $x'(t)$ is positive for large t.

Therefore, the only remaining possibility is $\lim_{t\to\infty} .x"(t) = 0$. If $\lim_{t\to\infty} P(t)$ is finite there is nothing to prove.

Now, suppose $\lim_{t\to\infty} P(t)$ is infinite and $\lim_{t\to\infty} .x"(t) = 0$.

By condition A1 it follows that for some constant λ

$$\lambda \leq -w(t) + K_0 + \int_T^t R'(s)\frac{x"(s)}{f[x(s)]}ds - \int_T^t R(s)\frac{x'(s)x"(s)f'[x(s)]}{f.^2[x(s)]}ds$$

Thus

$$-\int_T^t R'(s)\frac{x"(s)}{f[x(s)]}ds \leq -\lambda - R(t)\frac{x"(t)}{f[x(t)]} - \int_T^t R(s)\frac{x'(s)x"(s)f'[x(s)]}{f^2[x(s)]}ds$$

The last inequality gives

$$-R'(t)\frac{x'(t)}{f[x(t)]} - \int_T^t R'(s)\frac{x'^2(s)f'[x(s)]}{f^2[x(s)]}ds$$

$$\leq C_1 - R(t)\frac{x"(t)}{f[x(t)]} - \int_T^t R(s)\frac{x'(s)x"(s)f'[x(s)]}{f^2[x(s)]}ds$$

in the end

$$-R'(t)\frac{x'(t)}{f[x(t)]} + R(t)\frac{x"(t)}{f[x(t)]}$$

$$\leq C_1 + \int_T^t \left\{-R'(s)\frac{x'(s)}{f[x(s)]} + R(s)\frac{x"(s)}{f[x(s)]}\right\}[\frac{x'(s)f'[x(s)]}{f[x(s)]}]ds$$

Put

$$F(t) = -R'(t)\frac{x'(t)}{f[x(t)]} + R(t)\frac{x"(t)}{f[x(t)]} > 0$$

then

$$-F(t) \geq C_2 + \int_T^t F(s)(\frac{x'(s)f'[x(s)]}{f[x(s)]})ds \tag{A12}$$

where C_2 is constant. Furthermore, we choose a $T_1 \geq T$. So that

$$C_2 + \int_T^{T_1} F(s)(\frac{x'(s)f'[x(s)]}{f[x(s)]})ds = C_3 < 0 \tag{A13}$$

and then for every $t \geq T_1$ we get

$$F(t)\frac{x'(t)f'[x(t)]}{f[x(t)]} \geq \left[C_2 + \int_T^t F(s)(\frac{x'(s)f'[x(s)]}{f[x(s)]})ds\right](-\frac{x'(t)f'[x(t)]}{f[x(t)]})$$

this implies

$$F(t)\frac{x'(t)f'[x(t)]}{f[x(t)]}\left[C_2 + \int_T^t F(s)(\frac{x'(s)f'[x(s)]}{f[x(s)]})ds\right]^{-1} \leq (-\frac{x'(t)f'[x(t)]}{f[x(t)]})$$

and hence by integrating over $[T_1, t]$,we obtain for $t \geq T$

$$Log\frac{\left[C_2+\int_T^t F(s)(\frac{x'(s)f'[x(s)]}{f[x(s)]})ds\right]}{C_3} \leq Log\frac{f([x(T_1)}{f[x(t)]}$$

But,because of A13, we have

$$[C_2+\int_T^t F(s)(\frac{x'(s)f'(x(s)}{f[x(s)]})ds] \geq C_3\frac{f([x(T_1)}{f[x(t)]}$$

The previous inequality and A12 gives

$$R(t)x"(t) - R'(t)x'(t) \leq C_4$$

where $C_4 = C_3 f([x(T_1)]$.

But $\lim_{t\to\infty} R'(t)$ and $\lim_{t\to\infty} x'(t)$ are finite.Therefore, we conclude that $\lim_{t\to\infty} R(t)x"(t)$ is finite which establishes the theorem. ∎

3. Example

Example 1. *Consider the differential equation (E) with*
$q(t) = \frac{1}{3}t^{-4/3}(3+\cos(t)) - t^{1/3}\sin(t)$ *and* $R(t) = t, \quad t \geq T = \frac{\pi}{2}$
For every $t \geq T$*; we obtain*

$$\begin{aligned}
\int_T^t s.q(s)ds &= \int_{\pi/2}^t \left[\frac{1}{3}s^{-2/3}(3+\cos(s)) - s^{1/3}\sin(s)\right] ds \\
&= \int_{\pi/2}^t d(s^{1/3}(3+\cos(s)) \\
&= t^{1/3}(3+\cos(t) - 4(\frac{\pi}{2})^{1/3} \\
&\geq t^{1/3} - 4(\frac{\pi}{2})^{1/3}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{t}\int_T^t \int_{\pi/2}^s uq(u)ds &\geq \frac{1}{t}\int_T^t \left[s^{1/3} - 4(\frac{\pi}{2})^{1/3}\right] ds \\
&= \frac{3}{4}t^{1/3} - 4(\frac{\pi}{2})^{1/3} + \frac{13}{4t}(\frac{\pi}{2})^{4/3}
\end{aligned}$$

Thus, we have
$\liminf_{t\to\infty} \int_{\pi/2}^t sq(s)ds > -\infty$ *and* $\limsup_{t\to\infty} \frac{1}{t}\int_{\pi/2}^t \int_{\pi/2}^s uq(u)du.ds = \infty$
i.e. A1 and A2 are satisfed.Hence Theorem 1 and Theorem 2 guarantes that the set $\{x(t) \in \Pi \ ; \ x'(t).x"(t) > 0\} = \{x(t) \in \Pi \ ; \ x'(t) > 0, \ \ x"(t) < 0\} = \varnothing$.

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