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Introduction to b^{ϵ}-Near and b^{ϵ}_{δ}-Near Best Approximation in 2-Normed Space

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Abstract

In this research, "b^{ϵ}_{δ}-near best approximation" in the 2-normed space is defined. Also, it is shown that continuity of b^{ϵ}_{δ}-near best approximation with arbitrarily small ϵ and δ is enough to guarantee uniqueness in a strictly convex 2-normed space when the subset is boundedly compact and closed.

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1 Introduction

In [1] ϵ-near best approximation was defined, and some properties were studied. In this paper we introduce b^{ϵ}-near and b^{ϵ}_{δ}-near best approximation in 2-normed space and find some further results. first we need some definitions as follows,

Let X be a linear space of dimension greater than 1 over K, where K is the real or complex number field, and let $\| .,. \|$ be a real-valued function on $X \times X$ satisfying the following conditions:

- (i) $\| x, y \| = 0$ if and only if x and y are linearly dependent vectors.
- (ii) $\| x, y \| = \| y, x \|$ for all $x, y \in X$.
- (iii) $\| \lambda x, y \| = | \lambda | \| x, y \|$ for all $\lambda \in K$ and all $x, y \in X$.

- (iv) $\| x+y, z \| \leq \| x, z \| + \| y, z \|$for all $x, y, z \in X$.

Thus $\| .,. \|$ is called a 2-normed on X and $(X, \| .,. \|)$ is said to be a linear 2-normed space. Some of the basic properties of 2-norms are that they are non-negative and $\| x, y+\alpha x \| = \| x, y \|$ for all $x, y \in X$ and all $\alpha \in K$.

Let $(X, \| .,. \|)$ be a 2-normed space and $0 \neq b \in X$. The space X is strictly convex if and only if whenever x and y are distinct unit vectors all nontrivial convex combinations of the two have norm less than 1. For $x \in X$ and $r \geq 0$, let $B[x, r]$ denote the closed ball centered on x of radius r, with $bdB[x, r]$ its boundary sphere. For any subset A we write $cl(A)$ for its closure.

A subset M of X is boundedly compact if and only if the closure of $M \cap B$ is compact, for each closed ball B in X.

Let $(X, \| .,. \|)$ be a 2-normed space, $0 \neq b \in X$, G nonempty subset of X and $x \in X$. We define the functional $e_G : X \times < b > \longrightarrow R$ by

$$e_G(x) = \inf_{g \in G} \| x-g, b \| .$$

A set $G \subset X$ is called approximatively compact if for every $x \in X$ and every sequence $\{g_n\} \subset G$ with $\lim_{n\to\infty} \| x-g_n, b \| = e_G(x)$ there exists a subsequence $\{g_{n_k}\}$ converging to an element of G.

Suppose $\epsilon > 0$, we say that $g_0 \in G$ is the element of b^ϵ-near best approximation of x, if there exists $b \in X$ such that for each $g \in G$, we have

$$\| x-g_0, b \| \leq \| x-g, b \| + \epsilon \quad , \| x-g_0, b \| \neq 0.$$

We denote the set of all elements of b^ϵ-near best approximation of x by $P_G(x, b^\epsilon)$.

If for each $x \in X$, $P_G(x, b^\epsilon)$ is nonempty set (singleton, respectively), then G is called an b^ϵ-proximinal (b^ϵ-Chebyshev, respectively)subset of X.

More generally, suppose A is any nonempty subset of X. An b^ϵ-near best approximation of A by M is a map $\phi : A \to M$ such that

$$\| x-\phi(x), b \| \leq \| x-M, b \| + \epsilon$$

for all x in A.
Note that in this paper $\| x-y, b \|^\epsilon_\delta = (1+\epsilon) \| x-y \| + \delta$.

2 Main result

Definition 2.1 *Let $(X, \| .,. \|)$ be a 2-normed space, G subset of X, $b \in X$, $x \in X$, $\epsilon > 0$ and $\delta > 0$. We say that $g_0 \in G$ is the element of b^ϵ_δ-near best approximation of x, if there exists $b \in X$ such that for each $g \in G$, we have*

$$\| x-g_0, b \| \leq \| x-g, b \|^\epsilon_\delta \quad , \| x-g_0, b \| \neq 0.$$

We denote the set of all b^ϵ_δ-near best approximation of x, by $P_G(x, b^\epsilon_\delta)$.

Definition 2.2 *If for each x in 2-normed space $(X, \| .,. \|)$ and $b \in X$, $P_G(x, b_\delta^\epsilon)$ is nonempty set, then G is called a b_δ^ϵ- proximinal subset of X, and G is called a b_δ^ϵ- Chebyshev set, if for each $x \in X$, $P_G(x, b_\delta^\epsilon)$ is singleton.*

Theorem 2.3 *Let $(X, \| .,. \|)$ be a 2-normed space, $b \in X$, G subset of X, $x \in X \backslash (G + < b >)$, $g_0 \in G$, $\epsilon > 0$ and $\delta > 0$. We have $g_0 \in P_G(x, b_\delta^\epsilon)$ if and only if there exists an $f \in X_b^*$ with the following properties:*

$$\| f, b \| = 1, \qquad (I)$$

$$f(G) = 0 \qquad (II)$$

$$f(x - g_0) \geq \frac{\| x - g_0, b \| - \delta}{(1 + \epsilon)}. \qquad (III)$$

Proof: Assume that $g_0 \in P_G(x, b_\delta^\epsilon)$. Then since $x \in X \backslash (G + < b >)$, we have $\| x - g_0, b \| > 0$. Consequently, by virtue of a well known corollary of the Hahn-Banach theorem, there exists an $f \in X_b^*$ satisfying (I), (II) and $f(x - g_0) = e_G(x)$. Then,

$$\| x - g_0, b \| \leq (1 + \epsilon) e_G(x) + \delta = (1 + \epsilon) f(x - g_0) + \delta.$$

Therefore,

$$f(x - g_0) \geq \frac{\| x - g_0, b \| - \delta}{(1 + \epsilon)}.$$

Conversely, assume that there exists an $f \in X_b^*$ satisfying (I), (II) and (III). Then for any $g \in G$ we have

$$\begin{aligned} \| x - g_0, b \| &\leq (1 + \epsilon) f(x - g) + \delta \\ &\leq (1 + \epsilon) \mid f(x - g) \mid + \delta \\ &\leq (1 + \epsilon) \| f, b \| \| x - g, b \| + \delta \\ &= (1 + \epsilon) \| x - g, b \| + \delta. \end{aligned}$$

Therefore,

$$\| x - g_0, b \| \leq \inf_{g \in G} \| x - g, b \|_\delta^\epsilon,$$

whence $g_0 \in P_G(x, b_\delta^\epsilon)$, which completes the proof.

Definition 2.4 *Let $(X, \| .,. \|)$ be a 2-normed space, $b \in X$, M, A subsets of X, $\epsilon > 0$ and $\delta > 0$. An b_δ^ϵ-near best approximation of A by M is a map $\phi : A \to M$ such that*

$$\| x - \phi(x), b \| \leq \| x - M, b \|_\delta^\epsilon,$$

when, $x \in A$.

Theorem 2.5 *Let $(X, \| ., . \|)$ be a strictly convex 2-normed space, $b \in X$, $A \subset X$ and M be a closed, boundedly compact subset of X. Suppose that for each $\epsilon > 0$ and $\delta > 0$, there exists a b_δ^ϵ-near best approximation $\phi : A \to M$ of A by M. Then M is a b_δ^ϵ-Chebyshev set.*

Proof: Since M is boundedly compact and closed, any sequence $\{m_n\}$ in M with $lim_{n\to\infty} \| x - m_n, b \| = \| x - M, b \|$ accumulates at a point m in M, and so $P_M(x, b_\delta^\epsilon)$ is nonempty. It is sufficient to prove that $P_M(x, b_\delta^\epsilon)$ is a singleton set. See [1].

The next corollary gives conditions under which the existence of a single continuous b_δ^ϵ-near best approximation is sufficient to guarantee continuous unique best approximation.

Corollary 2.6 *Let $(X, \| ., . \|)$ be a strictly convex 2-normed space, $b \in X$ and let M be a closed, boundedly compact, positively homogeneous subset of X. Suppose that for some $\epsilon > 0$ and $\delta > 0$, there exists a continuous b_δ^ϵ-near best approximation $\phi : X \to M$ of X by M. Then M is a b_δ^ϵ-Chebyshev set.*

Proof: For $\lambda > 0$, consider the maps ϕ_λ by $\phi_\lambda(x) = \lambda\phi(\frac{x}{\lambda})$ for x in X. The map ϕ_λ is continuous. It is a $\lambda b_\delta^\epsilon$-near best approximation of X by M since

$$\begin{aligned} \| \phi_\lambda(x) - x, b \| &= \lambda \| \phi(\tfrac{x}{\lambda}) - \tfrac{x}{\lambda}, b \| \\ &\leq \lambda(1+\epsilon) \| \tfrac{x}{\lambda} - M, b \| + \lambda\delta \\ &= (1+\epsilon) \| x - M, b \| + \delta, \end{aligned}$$

for x in X. Previous theorem can be applied to this family of maps.

3 $b.b_\delta^\epsilon$-proximinality

Definition 3.1 *Let $(X, \| ., . \|)$ be 2-normed space. A closed subspace $Y \subset X$ for given $\epsilon > 0$, is said to be b_δ^ϵ-ball proximinal or simply $b.b_\delta^\epsilon$-proximinal if Y_1 is b_δ^ϵ-proximinal in X, where Y_1 is the unit ball of Y. Note that in this paper $\| x - y, b \|_\delta^\epsilon = (1+\epsilon) \| x - y \| + \delta$.*

Let $(X, \| ., . \|)$ be a 2-normed space and $Y \subset X$ be a closed subspace. We first show that for any $\epsilon > 0$, b_δ^ϵ-proximinality between the unit balls implies b_δ^ϵ-proximinality.

Lemma 3.2 *If the unit ball Y_1 for any $\epsilon > 0$, is b_δ^ϵ-proximinal in X_1 then Y is b_δ^ϵ-proximinal in X.*

Proof : Let $x \in X$ and $\alpha = \| x - Y, b \|_\delta^\epsilon$, for any $\epsilon > 0$. It is easy to see that

$$\alpha = \inf\{\| x - y, b \|_\delta^\epsilon : \| x - y, b \|_\delta^\epsilon \leq \alpha + \lambda\},$$

for any $\lambda > 0$.

Now let $\beta = \alpha + \lambda + \parallel x, b \parallel^{\epsilon}_{\delta}$. Since βY_1 is b^{ϵ}_{δ}-proximinal in βX_1, and as $x \in \beta X_1$, $\parallel x - \beta Y_1, b \parallel^{\epsilon}_{\delta} = \parallel x - y_0, b \parallel^{\epsilon}_{\delta}$ for some $y_0 \in \beta Y_1$.

For any $y \in Y$, $\parallel x - y, b \parallel^{\epsilon}_{\delta} \leq \alpha + \lambda$, we have $\parallel y, b \parallel^{\epsilon}_{\delta} \leq \beta$ so that $\parallel x - y, b \parallel^{\epsilon}_{\delta} \geq \parallel x - y_0, b \parallel^{\epsilon}_{\delta}$. Hence, $\parallel x - Y, b \parallel^{\epsilon}_{\delta} = \parallel x - y_0, b \parallel^{\epsilon}_{\delta}$.

Let Y be a b^{ϵ}_{δ}-proximinal. Let $P_{Y_1} : X \to 2^{Y_1}$ defined by $P_{Y_1}(x) = \{y \in Y_1 : \parallel x - Y_1, b \parallel^{\epsilon}_{\delta} = \parallel x - y, b \parallel^{\epsilon}_{\delta}\}$, denote the metric projection.

In the next proposition we collect several properties of the metric projection.

Lemma 3.3 *(1) For all $x \in X$, $\lambda > 0$, $\lambda P_{Y_1}(\frac{x}{\lambda}, b^{\epsilon}_{\delta}) = P_{\lambda Y_1}(x, b^{\epsilon}_{\delta})$.*
(2) $P_Y(x, b^{\epsilon}_{\delta}) = P_{\lambda Y_1}(x, b^{\epsilon}_{\delta})$ for all $\lambda \geq \parallel x, b \parallel^{\epsilon}_{\delta} + \parallel x - Y, b \parallel^{\epsilon}_{\delta}$.
(3) If P_{Y_1} is continuous, then P_Y is continuous.

Proof : (1) Let $x \in X$, $\lambda > 0$, let $y_0 \in P_{Y_1}(x, b^{\epsilon}_{\delta})$. For any $y \in Y_1$,

$$\parallel \frac{x}{\lambda} - y_0, b \parallel^{\epsilon}_{\delta} \leq \parallel \frac{x}{\lambda} - y, b \parallel^{\epsilon}_{\delta} \Leftrightarrow \parallel x - \lambda y_0, b \parallel^{\epsilon}_{\delta} \leq \parallel x - \lambda y, b \parallel^{\epsilon}_{\delta} .$$

Hence $\lambda P_{Y_1}(\frac{x}{\lambda}, b^{\epsilon}_{\delta}) = P_{\lambda Y_1}(x, b^{\epsilon}_{\delta})$.
(2) Now let $\lambda \geq \parallel x, b \parallel^{\epsilon}_{\delta} + \parallel x - Y, b \parallel^{\epsilon}_{\delta}$. For $y \in P_Y(x, b^{\epsilon}_{\delta})$, $\parallel y, b \parallel^{\epsilon}_{\delta} \leq \parallel x, b \parallel^{\epsilon}_{\delta} + \parallel x - y, b \parallel^{\epsilon}_{\delta} = \parallel x, b \parallel^{\epsilon}_{\delta} + \parallel x - Y, b \parallel^{\epsilon}_{\delta} \leq \lambda$. Hence $P_Y(x, b^{\epsilon}_{\delta}) \subset P_{\lambda Y_1}(x, b^{\epsilon}_{\delta})$. Since $\parallel x - Y, b \parallel^{\epsilon}_{\delta} \leq \parallel x - \lambda Y_1, b \parallel^{\epsilon}_{\delta}$, we have $P_Y(x, b^{\epsilon}_{\delta}) = P_{\lambda Y_1}(x, b^{\epsilon}_{\delta})$.
(3) Since P_{Y_1} is continuous if and only if $P_{\lambda Y_1}$ is continuous. By (2), P_Y is continuous.

Definition 3.4 *Let $(X, \parallel .,. \parallel)$ be 2-normed space. A closed $b.b^{\epsilon}_{\delta}$-proximinal subspace $Y \subset X$ is said to be strong $b.b^{\epsilon}_{\delta}$-proximinal at a point $x \in X$ if given $\epsilon' > 0$, there exists a $\delta' > 0$ such that $y \in Y_1$, $\parallel x - y, b \parallel^{\epsilon}_{\delta} \leq \parallel x - Y_1, b \parallel^{\epsilon}_{\delta} + \delta'$ implies there exists a $y' \in P_{Y_1}(x, b^{\epsilon}_{\delta})$ such that $\parallel y - y', b \parallel^{\epsilon}_{\delta} \leq \epsilon'$. It is said to be strong b^{ϵ}-proximinal if it is strong $b.b^{\epsilon}_{\delta}$-proximinal at each point.*

Lemma 3.5 *If Y is strong $b.b^{\epsilon}_{\delta}$-proximinal in X then it is strongly b^{ϵ}_{δ}-proximinal in X.*

Proof : we first show that Y_1 is strong b^{ϵ}_{δ}-proximinal if and only if λY_1 is strong b^{ϵ}_{δ}-proximinal for any $\lambda > 0$. Suppose Y_1 is strong b^{ϵ}_{δ}-proximinal. Fix $\lambda > 0$. Let $x \in X$. Since Y_1 is strong b^{ϵ}_{δ}-proximinal at $\frac{x}{\lambda}$, for $\epsilon' > 0$, there exists a $\delta' > 0$ such that for all $y \in Y_1$,

$$\parallel \frac{x}{\lambda} - y, b \parallel^{\epsilon}_{\delta} \leq \parallel \frac{x}{\lambda} - Y_1, b \parallel^{\epsilon}_{\delta} + \delta'$$

implies there exists $y' \in P_{Y_1}(\frac{x}{\lambda}, b_\delta^\epsilon)$ with $\| y - y', b \|_\delta^\epsilon \leq \frac{\epsilon'}{\lambda}$.
Let $\delta'' = \lambda\delta'$. If $y \in \lambda Y_1$ and $\| x - y, b \|_\delta^\epsilon \leq \| x - \lambda Y_1, b \|_\delta^\epsilon + \delta''$, then $\| \frac{x}{\lambda} - \frac{y}{\lambda}, b \|_\delta^\epsilon \leq \| \frac{x}{\lambda} - Y_1, b \|_\delta^\epsilon + \delta'$. Now let $y' \in P_{Y_1}(\frac{x}{\lambda}, b_\delta^\epsilon)$ be such that $\| y' - \frac{x}{\lambda}, b \|_\delta^\epsilon \leq \frac{\epsilon'}{\lambda}$. Thus $\| \lambda y' - y, b \|_\delta^\epsilon \leq \epsilon'\lambda$. As $y' \in P_{Y_1}(\frac{x}{\lambda}, b_\delta^\epsilon)$ implies that $\lambda y' \in P_{\lambda Y_1}$ we get the desired conclusion.
Finally if Y is strong $b.b_\delta^\epsilon$-proximinal at x, let $\lambda \geq \| x, b \|_\delta^\epsilon + \| x - Y, b \|_\delta^\epsilon$. By preceding lemma we have $P_{\lambda Y_1}(x) = P_Y(x)$. Hence Y is strong b_δ^ϵ-proximinal at x.

Lemma 3.6 *Let $(X, \| .,. \|)$ be 2-normed space $Z \subset Y \subset X$ be closed subspaces such that $\frac{Y}{Z}$ is b_δ^ϵ-proximinal in $\frac{X}{Z}$. If Z is b_δ^ϵ-proximinal in X then Y is b_δ^ϵ-proximinal in X.*

Proof : Let $x \in X$. Suppose $\| \pi(x) - \frac{Y}{Z}, b \|_\delta^\epsilon = \| \pi(x) - \pi(y_0), b \|_\delta^\epsilon$ for some $y_0 \in Y$. Now $\| \pi(x) - \pi(y_0), b \|_\delta^\epsilon = \| \pi(x - y_0), b \|_\delta^\epsilon = \| x - y_0 - Z, b \|_\delta^\epsilon = \| x - y_0 - z_0, b \|_\delta^\epsilon$ for some $z_0 \in Z$ as Z is b_δ^ϵ-proximinal in X. For $y \in Y$,

$$\| x - y, b \|_\delta^\epsilon \geq \| \pi(x - y), b \|_\delta^\epsilon \geq \| x - (y_0 + z_0), b \|_\delta^\epsilon .$$

As $y_0 + z_0 \in Y$ we get that Y is b_δ^ϵ-proximinal in X.

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