Positive Solutions for Second-Order Singular Semipositone Boundary Value Problems

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Abstract
In this paper, by using the fixed point index theorem, the existence results of positive solutions for some second-order singular semipositone boundary value problem are obtained.

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1 Introduction
Consider the following boundary value problem (BVP):

\[
\begin{cases}
    u'' + \frac{1}{\sqrt{t(1-t)(2+u)}} - \frac{1}{\sqrt{t(1-t)}}(2 - \frac{1}{e^u - 1}) = 0, & 0 < t < 1, \\
    au(0) - bu'(0) = 0, & cu(1) + du'(1) = 0,
\end{cases}
\]

which arises in many different areas of applied mathematics and physics. Singular problems of this type that the nonlinearity \(g\) may change sign are referred to as singular semipositone problems in the literature. Motivated by BVP (1.1), this paper presents the existence results of the following second-order singular semipositone boundary value problem:

\[
\begin{cases}
    u'' + f(t, u) + g(t, u) = 0, & 0 < t < 1, \\
    au(0) - bu'(0) = 0, & cu(1) + du'(1) = 0,
\end{cases}
\]

where \(a, b, c, d > 0\), \(f, g\) is singular at \(t = 0, 1\), and the sign of \(g\) may change.

In recent years, singular boundary value problems in the case of \(g \equiv 0\) (i.e. positone problems) have been studied extensively (see, for example, [1-4] and references therein). Naturally we hope there are the same excellent results on singular semipositone boundary value problems, so the purpose of this paper is to deal with the second-order singular semipositone boundary value problems.
2 Preliminary Notes

Concerning the boundary value problem (1.2), we make the following hypotheses:

\((H_1)\) \(f, g : (0, 1) \times (0, \infty) \rightarrow R\) is continuous \(f(t, u) \geq 0, \ g(t, u) > -q(t), \ \forall (t, u) \in (0, 1) \times (0, \infty), \) where \(q(t) \in L^1(0, 1), \ \) \(q(t) \geq 0;\)

\((H_2)\) \(f(t, u) \leq p_1(t)q_1(u), \ g(t, u) \leq p_2(t)q_2(u), \) where \(p_1 \in C[[0, 1) \rightarrow R^+], \ p_2 \in C[[0, 1) \rightarrow R], \ q_1 \in C[R^+ \rightarrow R^+], \ q_2 \in C[R^+ \rightarrow R], \ 0 < \int_0^1 e(s)q(s)ds < \infty, \) and \(0 < \int_0^1 e(s)(p_1(s) + p_2(s) + q(s))ds < \infty, \) where \(e(s) = \frac{(b+as)(d+c(1-s))}{ac+ad+bc};\)

\((H_3)\) There exists \(r : r > 2 \int_0^1 q(s)ds\) such that

\[
\max_{0 \leq ||u|| \leq r} \left\{ q_1(u), |q_2(u)|, 1 \right\} > \int_0^1 e(s)(p_1(s) + p_2(s) + q(s))ds;
\]

\((H_4)\) \(\lim_{u \rightarrow -\infty} \inf_{0 < t < 1} \frac{f(t,u)}{u} > M, \) where \(M^{-1} = \frac{1}{2} \max_{0 \leq t \leq 1} \int_0^1 G(t,s)ds, \ \theta \in (0, \frac{1}{2}).\)

We define \(E = C[0,1]\) be the Banach space, \(||u|| = \max_{0 \leq t \leq 1} |u(t)|\) be the norm of \(E,\) and \(G(t,s)\) be the Green's function of the following second-order boundary value problem:

\[
\begin{align*}
&u'' = 0, \quad 0 < t < 1, \\
&au(0) - bu'(0) = 0, \quad cu(1) + du'(1) = 0,
\end{align*}
\]

that is

\[
G(t, s) = \begin{cases}
\frac{(b+at)(d+c(1-s))}{ac+ad+bc}, & 0 \leq t \leq s \leq 1, \\
\frac{(b+as)(d+c(1-t))}{ac+ad+bc}, & 0 \leq s \leq t \leq 1.
\end{cases}
\]

It is clearly that: \(G(t, s) \geq 0, \ G(t, s) \leq e(t) = \frac{(b+at)(d+c(1-t))}{ac+ad+bc}\) for \(\forall (t, s) \in (0,1).\) We define a cone \(K \subset E\) by

\[
K = \{ u \mid u \in E, \ u(t) \geq e(t)||u||, \ \text{for} \ \ t \in [0,1] \}.
\]

Let

\[
[u]^* = \begin{cases}
u, u \geq 0, \\
o, u < 0,
\end{cases}
\]

Let us define a mapping \(A : K \rightarrow E\) by

\[
Au(t) = \int_0^1 G(t, s)(f(s,[u-\omega]^*) + g(s,[u-\omega]^*) + q(s))ds,
\]

where

\[
\omega(t) = \int_0^1 G(t, s)q(s)ds.
\]
It is easy to check that $u$ is the positive solution of the second-order singular semipositone boundary value problem (1.2) if and only if $u > 0$ and $u + \omega$ is the fixed point of $A$.

The following lemmas are important to the proof of our theorem.

**Lemma 2.1** [5] Let $K$ be a cone of the real Banach space $E, \Omega$ be a bounded set of $E, \theta \in \Omega$, and $A: \bar{\Omega} \cap K \to K$ be completely continuous. Suppose that $Au \neq m Bu$ for $\forall u \in \partial \Omega \cap K, m \geq 1$, then $i(A, \bar{\Omega} \cap K, K) = 1$.

**Lemma 2.2** [5] Let $K$ be a cone of the real Banach space $E, \Omega$ be a bounded set of $E, \theta \in \Omega$, and $A: \bar{\Omega} \cap K \to K$ be completely continuous. Suppose that there exists $B: \partial \Omega \cap K \to K$ which is completely continuous such that

1. $\inf_{u \in \partial \Omega \cap K} \|Bu\| > 0$,
2. $u - Au \neq m Bu, \forall u \in \partial \Omega \cap K, m \geq 0$,

then $i(A, \bar{\Omega} \cap K, K) = 0$.

**Lemma 2.3** If $(H_1), (H_2)$ hold, then $A: K \to K$ is completely continuous.

**Proof** We define a function $\tau(s) : [0, 1] \to [0, 1]$ such that $G(\tau(s), s) = \max_{0 \leq t \leq 1} G(t, s)$, then we have

$$\frac{G(t, s)}{G(\tau(s), s)} = \begin{cases} \frac{b + at}{b + a\tau(s)}, & t, \tau(s) \leq s, \\ \frac{b + at)(d + c(1-s))}{(b + a\tau(s))(d + c(1-s))}, & t \leq s \leq \tau(s), \\ \frac{d + c(1 - \tau(s))}{(b + a\tau(s))}, & s \leq t, \tau(s), \\ \frac{d + c(1 - \tau(s))}{(b + a\tau(s))(d + c(1-s))}, & \tau(s) \leq s \leq t \end{cases}$$

thus, for $\forall u \in K$, we get

$$Au(t) = \int_0^1 G(t, s)(f(s, ([u - \omega]^*)) + g(s, ([u - \omega]^*)) + q(s))ds$$

$$= \int_0^1 \frac{G(t, s)}{G(\tau(s), s)}G(\tau(s), s)(f(s, ([u - \omega]^*)) + g(s, ([u - \omega]^*)) + q(s))ds$$

$$\geq e(t)\|Au\|, \quad t \in [0, 1],$$

which implies $A : K \to K$, and it is easy to prove that $A$ is completely continuous (see [6]).

### 3 Main Results

**Theorem 3.1** Suppose $(H_1) - (H_4)$ hold, then BVP (1.2) has at least one positive solution.

**Proof** Firstly, we prove: $u \neq \alpha Au, \alpha \in [0, 1], \ u \in \partial \Omega_r$, $\Omega_r = \{u \in \Omega \mid \|u\| < r\}$. 


If otherwise, there exist $\alpha_0 \in [0, 1]$, $u_0 \in \partial \Omega_r$ such that

$$u_0 = \alpha_0 Au_0,$$

then we have

$$u_0(t) \geq \|u_0\|e(t) = r \cdot e(t),$$
on the other hand,

$$\omega(t) = \int_0^1 G(t, s)q(s)ds \leq \int_0^1 q(s)ds \cdot e(t) \leq \frac{\int_0^1 q(s)ds}{r}u_0(t)ds, \ t \in [0, 1]. \quad (3.1)$$

Thus

$$u_0(t) - \omega(t) \geq (1 - \frac{\int_0^1 q(s)ds}{r})u_0(t) \geq \frac{1}{2}u_0(t) \geq 0, \ t \in [0, 1], \quad (3.2)$$

and so

$$u_0 = \alpha_0 Au_0$$

$$= \alpha_0 \int_0^1 G(t, s)(f(s, [u_0 - \omega]^*) + g(s, [u_0 - \omega]^*) + q(s))ds$$

$$\leq \int_0^1 G(t, s)(f(s, [u_0 - \omega]^*) + g(s, [u_0 - \omega]^*) + q(s))ds$$

$$\leq \int_0^1 G(t, s)(p_1(s)q_1([u_0 - \omega]) + p_2(s)q_2([u_0 - \omega]) + q(s))ds$$

$$\leq \int_0^1 e(s)(p_1(s) + p_2(s) + q(s))ds \cdot \max_{0 \leq u \leq r} \{q_1(u), q_2(u), 1\}$$

then $\|u_0\| = r \leq \int_0^1 e(s)(p_1(s) + p_2(s) + q(s))ds \cdot \max_{0 \leq u \leq r} \{q_1(u), q_2(u), 1\}$, which is contradict to $(H_3)$. By Lemma 2.1 we know

$$i(A, \Omega_r \cap K, K) = 1. \quad (3.3)$$

By $(H_4)$, for $\forall \varepsilon > 0$, there exists $R' > r > 0$ such that $f(t, u) > (M + \varepsilon)u$ for $\forall \varepsilon > 0, u > R'$. Let $\Omega_R = \{u \in \Omega | \|u\| < R\}$, where $R \geq 2^{\frac{(b+a)(d+\varphi)}{(b+\alpha)(d+\varphi)}}R'$. Let $Bu(t) \equiv 1$, for $\forall t \in [0, 1]$. It is clearly that $\inf_{u \in \partial \Omega_R \cap K} \|Bu\| > 0$, and

$$B : \partial \Omega \cap K \to K$$

is completely continuous.

Secondly, we prove that $u - Au \not\equiv mBu$ for $\forall u \in \partial \Omega \cap K$, $m \geq 0$.

Otherwise, there exist $v_0 \in \partial \Omega_R, m_0 \geq 0$, such that $v_0 - Av_0 = m_0 Bu_0$. Let $\xi = \min\{v_0(t), t \in [\theta, 1 - \theta], \theta \in (0, \frac{1}{2})\}$. Since $R > r > 0$, noticing that (3.2) holds, then for any $t \in [\theta, 1 - \theta]$, we have

$$v_0(t) = \int_0^1 G(t, s)(f(s, [v_0 - \omega]^*) + g(s, [v_0 - \omega]^*) + q(s))ds + m_0$$

$$\geq \int_0^{1-\theta} G(t, s)(f(s, [v_0 - \omega])ds + m_0$$

$$\geq \int_0^{1-\theta} G(t, s)\frac{1}{2}(M + \varepsilon)ds \cdot \xi + m_0$$

$$= M^{-1}(M + \varepsilon)\xi + m_0$$

$$> \xi,$$
which is contradicts to \( \xi = \min \{ v_0(t), t \in [\theta, 1 - \theta], \theta \in (0, \frac{1}{2}) \} \). Then for
\( \forall u \in \partial \Omega_R \cap K, m \geq 0 \), we have \( u - Au \neq mBu \). By Lemma 2.2, we get
\[
i(A, \Omega_R \cap K, K) = 0. \tag{3.4}
\]

It follows from (3.3) – (3.4) that \( i(A, (\overline{\Omega_R} \setminus \Omega_r) \cap K, K) = -1 \). According
to the fixed point index theorem (see[7]), we get that \( A \) has a fixed point \( \bar{u}, r \leq \| \bar{u} \| \leq R \). Let \( v = \bar{u} - \omega \). Since \( \| \bar{u} \| \geq r \), then \( \bar{u}(t) - \omega(t) \geq (1 - \frac{\int_0^1 q(s) ds}{r}) \bar{u}(t) \geq \frac{1}{2} \bar{u}(t) \geq 0 \). Thus \( v = \bar{u} - \omega \) is a positive solution of BVP (1.2).

In BVP (1.1), let \( f(t, u) = \frac{1}{\sqrt{t(1-t)(2+u)}} \), \( g(t, u) = -\frac{1}{\sqrt{t(1-t)}}(2 - \frac{1}{e^u - 1}) \), \( q(t) = \frac{2}{\sqrt{t(1-t)}} \), \( p_1(t) = \frac{1}{\sqrt{t(1-t)}} \), \( q_1(u) = \frac{1}{1+u} \), \( p_2(t) = -\frac{1}{\sqrt{t}} \), \( q_2(u) = 2 - \frac{1}{e^u - 1} \).

It is easy to check that \( (H_1), (H_3) \) hold. Let \( r = 14 \), then \( r > 2 \int_0^1 q(s) ds > 2 \int_0^1 \frac{2}{\sqrt{t(1-t)}} ds = 4\pi, \max_{0 \leq s \leq 7} \frac{r}{\int_0^1 q_1(u)} > 7 > \int_0^1 s(1-s)(p_1(s) + p_2(s) + q(s)) ds = \frac{1}{6\sqrt{6}} + \frac{4}{63} \), then \( (H_2) \) also holds. It follows from Theorem 1 that
BVP (1.1) has at least one positive solution.

References


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