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Positive Solutions for Second-Order Singular Semipositone Boundary Value Problems

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Abstract

In this paper,by using the fixed point index theorem,the existence results of positive solutions for some second-order singular semipositone boundary value problem are obtained.

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1 Introduction

Consider the following boundary value problem(BVP):

$$\begin{cases} u^{''} + \frac{1}{\sqrt{t}(1-t)(2+u)} - \frac{1}{\sqrt{t(1-t)}}(2 - \frac{1}{e^u - 1}) = 0, 0 < t < 1, \\ au(0) - bu^{'}(0) = 0, \ cu(1) + du^{'}(1) = 0, \end{cases} \tag{1.1}$$

which arises in many different areas of applied mathematics and physics. Singular problems of this type that the nonlinearity g may change sign are referred to as singular semipositone problems in the literature. Motivated by BVP (1.1), this paper presents the existence results of the following second-order singular semipositone boundary value problem:

$$\begin{cases} u^{''} + f(t,u) + g(t,u) = 0, \quad 0 < t < 1, \\ au(0) - bu^{'}(0) = 0, cu(1) + du^{'}(1) = 0, \end{cases} \tag{1.2}$$

where $a, b, c, d > 0$, f, g is singular at $t = 0, 1$, and the sign of g may change.

In recent years, singular boundary value problems in the case of $g \equiv 0$ (i.e.positone problems) have been studied extensively(see,for example,[1-4]and references therein). Naturally we hope there are the same excellent results on singular semipositone boundary value problems, so the purpose of this paper is to deal with the second-order singular semipositone boundary value problems.

2 Preliminary Notes

Concerning the boundary value problem(1.2), we make the following hypotheses:

(H_1) $f,\ g:(0,1)\times(0,\infty)\to R$ is continuous$f(t,u)\geq 0,\quad g(t,u)>-q(t),\ \forall(t,u)\in(0,1)\times(0,\infty)$, where $q(t)\in L^1(0,1),\quad q(t)\geq 0$;

(H_2) $f(t,u)\leq p_1(t)q_1(u),\ g(t,u)\leq p_2(t)q_2(u)$, where $p_1\in C[(0,1)\to R^+],\ p_2\in C[(0,1)\to R],\ q_1\in C[R^+\to R^+],\ q_2\in C[R^+\to R],\ 0<\int_0^1 e(s)q(s)ds<\infty$, and $0<\int_0^1 e(s)(p_1(s)+p_2(s)+q(s))ds<\infty$, where $e(s)=\frac{(b+as)(d+c(1-s))}{ac+ad+bc}$;

(H_3) There exists $r:r>2\int_0^1 q(s)ds$ such that

$$\frac{r}{\max\limits_{0\leq\|u\|\leq r}\{q_1(u),|q_2(u)|,1\}}>\int_0^1 e(s)(p_1(s)+p_2(s)+q(s))ds;$$

(H_4) $\underline{\lim}_{u\to\infty}\inf\limits_{0<t<1}\frac{f(t,u)}{u}>M$, where $M^{-1}=\frac{1}{2}\max\limits_{0\leq t\leq 1}\int_\theta^{1-\theta}G(t,s)ds,\ \theta\in(0,\frac{1}{2})$.

We define $E=C[0,1]$ be the Banach space, $\parallel u\parallel=\max\limits_{0\leq t\leq 1}|u(t)|$ be the norm of E,and G(t,s) be the Green's function of the following second-order boundary value problem:

$$\begin{cases} u''=0,\quad 0<t<1,\\ au(0)-bu'(0)=0,\quad cu(1)+du'(1)=0,\end{cases}$$

that is

$$G(t,s)=\begin{cases}\frac{(b+at)(d+c(1-s))}{ac+ad+bc},\ 0\leq t\leq s\leq 1,\\ \frac{(b+as)(d+c(1-t))}{ac+ad+bc},\ 0\leq s\leq t\leq 1.\end{cases}$$

It is clearly that: $G(t,s)\geq 0,\quad G(t,s)\leq e(t)=\frac{(b+at)(d+c(1-t))}{ac+ad+bc}$ for $\forall\ (t,s)\in(0,1)$. We define a cone $K\subset E$ by

$$K=\{u\mid u\in E,\ u(t)\geq e(t)\|u\|,\quad t\in[0,1]\}.$$

Let

$$[u]^*=\begin{cases}u,u\geq 0,\\ 0,u<0,\end{cases}$$

Let us define a mapping $A:K\to E$ by

$$Au(t)=\int_0^1 G(t,s)(f(s,[u-\omega]^*)+g(s,[u-\omega]^*)+q(s))ds,$$

where

$$\omega(t)=\int_0^1 G(t,s)q(s)ds.$$

It is easy to check that u is the positive solution of the second-order singular semipositone boundary value problem (1.2) if and only if $u > 0$ and $u + \omega$ is the fixed point of A .

The following lemmas are important to the proof of our theorem.

Lemma 2.1[5] Let K be a cone of the real Banach space E, Ω be a bounded set of E, $\theta \in \Omega$, and $A : \overline{\Omega} \cap K \to K$ be completely continuous. Suppose that $Au \neq mu$ for $\forall\ u \in \partial\Omega \cap K,\ m \geq 1$, then $i(A, \Omega \cap K, K) = 1$.

Lemma 2.2[5] Let K be a cone of the real Banach space E, Ω be a bounded set of E, $\theta \in \Omega$, and $A : \overline{\Omega} \cap K \to K$ be completely continuous. Suppose that there exists $B : \partial\Omega \cap K \to K$ which is completely continuous such that

(1) $\inf\limits_{u \in \partial\Omega \cap K} \|Bu\| > 0$,

(2) $u - Au \neq mBu, \forall u \in \partial\Omega \cap K,\ m \geq 0$,

then $i(A, \Omega \cap K, K) = 0$.

Lemma 2.3 If $(H_1), (H_2)$ hold, then $A : K \to K$ is completely continuous.

Proof We define a function $\tau(s) : [0,1] \to [0,1]$ such that $G(\tau(s), s) = \max\limits_{0 \leq t \leq 1} G(t,s)$, then we have

$$\frac{G(t,s)}{G(\tau(s),s)} = \begin{cases} \frac{b+at}{b+a\tau(s)}, & t, \tau(s) \leq s, \\ \frac{(b+at)(d+c(1-s))}{(b+as)(d+c(1-\tau(s)))}, & t \leq s \leq \tau(s), \\ \frac{d+c(1-t)}{d+c(1-\tau(s))}, & s \leq t, \tau(s), \\ \frac{(b+as)(d+c(1-t))}{(b+a\tau(s))(d+c(1-s))}, & \tau(s) \leq s \leq t \end{cases}$$

$$\geq e(t) = \frac{(b+at)(d+c(1-t))}{ac+ad+bc},$$

thus, for $\forall\ u \in K$, we get

$$\begin{aligned} Au(t) &= \textstyle\int_0^1 G(t,s)(f(s,([u-\omega]^*)) + g(s,([u-\omega]^*)) + q(s))ds \\ &= \textstyle\int_0^1 \frac{G(t,s)}{G(\tau(s),s)} G(\tau(s),s)(f(s,([u-\omega]^*)) + g(s,([u-\omega]^*)) + q(s))ds \\ &\geq e(t)\|Au\|,\ t \in [0,1], \end{aligned}$$

which implies $A : K \to K$, and it is easy to prove that A is completely continuous(see [6]).

3 Main Results

Theorem 3.1 *Suppose* $(H_1) - (H_4)$ *hold, then BVP* (1.2) *has at least one positive solution.*

Proof Firstly, we prove: $u \neq \alpha Au, \alpha \in [0,1],\ u \in \partial\Omega_r,\ \Omega_r = \{u \in \Omega \mid \|u\| < r\}$.

If otherwise, there exist $\alpha_0 \in [0,1]$, $u_0 \in \partial\Omega_r$ such that

$$u_0 = \alpha_0 A u_0,$$

then we have

$$u_0(t) \geq \|u_0\| e(t) = r \cdot e(t),$$

on the other hand,

$$\omega(t) = \int_0^1 G(t,s)q(s)ds \leq \int_0^1 q(s)ds \cdot e(t) \leq \frac{\int_0^1 q(s)ds}{r} u_0(t)ds, \ t \in [0,1]. \quad (3.1)$$

Thus

$$u_0(t) - \omega(t) \geq (1 - \frac{\int_0^1 q(s)ds}{r}) u_0(t) \geq \frac{1}{2} u_0(t) \geq 0, \ t \in [0,1], \quad (3.2)$$

and so

$$\begin{aligned} u_0 &= \alpha_0 A u_0 \\ &= \alpha_0 \textstyle\int_0^1 G(t,s)(f(s,[u_0-\omega]^*) + g(s,[u_0-\omega]^*) + q(s))ds \\ &\leq \textstyle\int_0^1 G(t,s)(f(s,[u_0-\omega]^*) + g(s,[u_0-\omega]^*) + q(s))ds \\ &\leq \textstyle\int_0^1 G(t,s)(p_1(s)q_1([u_0-\omega]) + p_2(s)q_2([u_0-\omega]) + q(s))ds \\ &\leq \textstyle\int_0^1 G(t,s)(p_1(s) + p_2(s) + q(s))ds \cdot \max_{0\leq u\leq r}\{q_1(u), |q_2(u)|, 1\} \\ &\leq \textstyle\int_0^1 e(s)(p_1(s) + p_2(s) + q(s))ds \cdot \max\limits_{0\leq u\leq r}\{q_1(u), |q_2(u)|, 1\} \end{aligned}$$

then $\|u_0\| = r \leq \int_0^1 e(s)(p_1(s)+p_2(s)+q(s))ds \cdot \max\limits_{0\leq u\leq r}\{q_1(u), |q_2(u)|, 1\}$, which is contradict to (H_3). By Lemma 2.1 we know

$$i(A, \Omega_r \cap K, K) = 1. \quad (3.3)$$

By (H_4), for $\forall \varepsilon > 0$, there exists $R' > r > 0$ such that $f(t,u) > (M+\varepsilon)u$ for $\forall \varepsilon > 0, u > R'$. Let $\Omega_R = \{u \in \Omega \mid \|u\| < R\}$, where $R \geq 2\frac{(b+a)(d+c)}{(b+a\theta)(d+c\theta)}R'$. Let $Bu(t) \equiv 1$, for $\forall t \in [0,1]$. It is clearly that $\inf\limits_{u\in\partial\Omega_R\cap K} \|Bu\| > 0$, and $B : \partial\Omega \cap K \to K$ is completely continuous.

Secondly, we prove that $u - Au \neq mBu$ for $\forall u \in \partial\Omega \cap K$, $m \geq 0$.

Otherwise, there exist $v_0 \in \partial\Omega_R, m_0 \geq 0$, such that $v_0 - Av_0 = m_0 B v_0$. Let $\xi = \min\{v_0(t), t \in [\theta, 1-\theta], \theta \in (0, \frac{1}{2})$. Since $R > r > 0$, noticing that (3.2) holds, then for any $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned} v_0(t) &= \textstyle\int_0^1 G(t,s)(f(s,[v_0-\omega]^*) + g(s,[v_0-\omega]^*) + q(s))ds + m_0 \\ &\geq \textstyle\int_\theta^{1-\theta} G(t,s)(f(s,[v_0-\omega])ds + m_0 \\ &\geq \textstyle\int_\theta^{1-\theta} G(t,s)\frac{1}{2}(M+\varepsilon)ds \cdot \xi + m_0 \\ &= M^{-1}(M+\varepsilon)\xi + m_0 \\ &> \xi, \end{aligned}$$

which is contradicts to $\xi = \min\{v_0(t), t \in [\theta, 1-\theta], \theta \in (0, \frac{1}{2})\}$. Then for $\forall u \in \partial\Omega_R \cap K, m \geq 0$, we have $u - Au \neq mBu$. By Lemma 2.2, we get

$$i(A, \Omega_R \cap K, K) = 0. \tag{3.4}$$

It follows from (3.3) − (3.4) that $i(A, (\overline{\Omega_R} \setminus \Omega_r) \cap K, K) = -1$. According to the fixed point index theorem (see[7]), we get that A has a fixed point $\widetilde{u}$, $r \leq \|\widetilde{u}\| \leq R$. Let $v = \widetilde{u} - \omega$. Since $\|\widetilde{u}\| \geq r$, then $\widetilde{u}(t) - \omega(t) \geq (1 - \frac{\int_0^1 q(s)ds}{r})\widetilde{u}(t) \geq \frac{1}{2}\widetilde{u}(t) \geq 0$. Thus $v = \widetilde{u} - \omega$ is a positive solution of BVP (1.2).

In BVP (1.1), let $f(t,u) = \frac{1}{\sqrt{t}(1-t)(2+u)}$, $g(t,u) = -\frac{1}{\sqrt{t(1-t)}}(2 - \frac{1}{e^u - 1})$, $q(t) = \frac{2}{\sqrt{t(1-t)}}$, $p_1(t) = \frac{1}{\sqrt{t}(1-t)}$, $q_1(u) = \frac{1}{1+u}$, $p_2(t) = -\frac{1}{\sqrt{t}}$, $q_2(u) = 2 - \frac{1}{e^u - \frac{1}{2}}$.

It is easy to check that $(H_1), (H_3)$ hold. Let $r = 14$, then $r > 2\int_0^1 q(s)ds > 2\int_0^1 \frac{2}{\sqrt{t(1-t)}}ds = 4\pi$, $\frac{r}{\max\limits_{0 \leq u \leq r}\{q_1(u), |q_2(u)|, 1\}} > 7 > \int_0^1 s^2(1-s)^2(p_1(s) + p_2(s) + q(s))ds = \frac{1}{6\sqrt{6}} + \frac{4}{63}$, then (H_2) also holds. It follows from Theorem 1 that BVP (1.1) has at least one positive solution.

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