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On a Perturbed System of Chemotaxis

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Abstract

We consider the blow up mechanism for a perturbed system of chemotaxis. First, using Moser's iteration scheme the blow up point of the solution is characterized in terms of the local Zygmund norm. And then, by the analysis on the Gagliard-Nirenberg inequality together with the study of the Green's function associated with elliptic part of system, we establish finiteness of blow up point.

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1 Introduction

We consider the following parabolic-elliptic system describing chemotactic aggregation of slime molds:

$$\text{(CZ)} \quad \begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v), \qquad (x,t) \in \Omega \times (0,T) \\ 0 = \Delta v - \gamma v - \beta |v|^{p-1} v + \alpha u, \qquad (x,t) \in \Omega \times (0,T) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and $T > 0$. Here, $u(x,t)$ and $v(x,t)$ are the cell density and the concentration of the chemical substance at place x and time t, respectively, χ, α, β, γ are positive constants, $1 < p < \infty$, and ν is the unit outward normal vector on $\partial\Omega$. The term $F = \nabla u - \chi u \nabla v$ stands for the flux of u so that the effect of diffusion $\nabla \cdot \nabla u$ and that of chemotaxis $\chi \nabla \cdot (u \nabla v)$ are competing for u to vary.

For the problem (CZ) with $\beta = 0$, Nagai [7] showed that the conjecture of Childress and Percus [5] on $N = 2$ is true for radially symmetric case. More precisely, the chemotaxis collapse can occur if the total cell number on $\Omega \subset \mathbb{R}^2$, indicated by $\lambda = \|u_0\|_1$, is larger than $\frac{8\pi}{\alpha\chi}$ and can not occur in the other case. Then, Senba and Suzuki [11] and Suzuki [12] have made clear the blowup mechanism including non-radial case. Here we study the blowup solution in the case of $\beta \neq 0$. For the initial function u_0 we suppose

1. $u_0 \geq 0$ and u_0 is not identical to 0 on Ω,
2. u_0 is smooth on $\overline{\Omega}$.

Chen and Zhong [3] showed that system (CZ) has a unique classical solution denoted by $(u, v) = (u(x,t), v(x,t))$ defined for $(x,t) \in \overline{\Omega} \times [0, T_{\max})$ under the assumption of $1 < p < +\infty$ for $N = 2$ and $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, where $T_{\max} = \sup\{T > 0 \mid (u,v) \text{exists for } x \in \overline{\Omega}, t \in [0,T)\}$ stands for the maximal existence time of the solution. This solution satisfies $u \geq 0$ and $\lambda = \|u(\cdot,t)\|_1$ is invariant in t. Moreover, [4] obtained that if $N = 2$ and $\|u_0\|_1 = \lambda < \frac{4\pi}{\alpha\chi}$ then $T_{\max} = \infty$ and $\|u(\cdot,t)\|_\infty \leq C$, and also that if u_0 is radially symmetric and $\|u_0\|_1 > \frac{8\pi}{\alpha\chi}$, then $T_{\max} < \infty$ can happen.

Henceforth we assume $\chi = \alpha = \beta = \gamma = 1$ for simplicity. We shall show two theorems concerning the blowup of the solution. The first theorem justifies the terminology 'blowup'.

Theorem 1.1 *If* $T_{\max} < \infty$*, then* $\lim_{t \uparrow T_{\max}} \|u(\cdot,t)\|_\infty = \infty$.

Regarding this, we define the blowup set $\mathcal{B}$ of u as follows:

$$\mathcal{B} = \{x_0 \in \bar{\Omega} \mid \text{there exist } t_k \uparrow T_{\max} \text{ and } x_k \to x_0 \text{ such that } u(x_k, t_k) \to \infty \text{ as } k \to \infty\},$$

and call each $x_0 \in \mathcal{B}$ the blowup point. The condition $T_{\max} < \infty$ implies $\mathcal{B} \neq \emptyset$ by Theorem 1.1, while the next theorem guarantees its finiteness.

Theorem 1.2 *If* $T_{\max} < \infty$*, then* $\sharp\mathcal{B} < \infty$.

Keller and Segel [6] discussed the initiation of cell aggregation as instability of the spatially homogeneous steady state. As for the global in time behavior of the solution, Nanjundiah [10] has posed the problem that cell density $u(x,t)$ will blowup in finite time and form a δ-function singularity. Such a result is obtained for $\beta = 0$ by [11], and then the quantized blowup mechanism is proven by [12] suggested by the threshold conjectured of [5]. Even in this perturbed system of $\beta \neq 0$, we can show the formation of collapse by Theorem 1.2 similarly. Thus, it holds that $u(\cdot, t) \rightharpoonup \sum_{x_0 \in \mathcal{B}} m(x_0)\delta_{x_0}$ in $\mathcal{M}(\overline{\Omega})$ as $t \uparrow T_{\max}$. Here, mass quantization indicates the equality $m(x_0) = m_*(x_0)$, where $m_*(x_0) = 8\pi$ and $m_*(x_0) = 4\pi$ according to $x_0 \in \Omega$ and $x_0 \in \partial\Omega$, respectively. The estimate $m(x_0) \leq m_*(x_0)$ may be proven similarly to [12], while $m(x_0) \geq m_*(x_0)$ will be more delicate because there seems to be no (quasi-)Lyapunov function associated with the Trudinger-Moser inequality.

2 Preliminaries

In this section we confirm several inequalities used later. First, the Gagliardo-Nirenberg inequality in two space dimensions is described by

$$\|w\|_2^2 \leq K^2(\|\nabla w\|_1^2 + \|w\|_1^2), \qquad w \in W^{1,1}(\Omega),$$

where K is a constant determined by Ω.

Henceforth, we set $B_R(x_0) = \{x \in \mathbf{R}^2 \mid |x - x_0| < R\}$ and take the smooth cut-off function φ satisfying

$$0 \leq \varphi \leq 1 \quad \text{in } \mathbf{R^2}, \qquad \frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \tag{1}$$

as in [11, 12]. More precisely, given $x_0 \in \Omega$, we take $B_{2R}(x_0) \subset \Omega$ and $\varphi \in C_0^\infty(\mathbf{R}^2)$ satisfying $0 \leq \varphi \leq 1$ and where $R' \in (0, R)$. Given $x_0 \in \partial\Omega$, we take $0 < R \ll 1$ and the smooth conformal mapping $X : B_{2R}(x_0) \cap \bar{\Omega} \to \mathbf{R^2}$ satisfying $x_0 \mapsto 0$ and where $R' \in (0, R)$. Then we set $\varphi = \zeta(X(x))$, where $\zeta \in C_0^\infty(\mathbf{R}^2)$, $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B_{1/2}(0)$, and $\zeta = 0$ on $\mathbf{R}^2 \setminus B_1(0)$. It holds that

$$\frac{\partial}{\partial \nu} \zeta \circ X = \frac{\partial X}{\partial \nu} \cdot (\nabla \zeta \circ X) = 0 \quad \text{on } \partial\Omega,$$

because X is conformal and hence $\frac{\partial X}{\partial \nu}$ is proportional to $(0, 1)$ on $\partial\Omega$. Thus, we obtain (1) in both cases.

Then, $\psi = \varphi^6_{x_0,R',R}$ satisfies where $A > 0$, $B > 0$ are constants determined by $0 < R' < R \ll 1$. We shall use the following lemma obtained by [11].

Lemma 2.1 *The following inequalities hold for any $s > 1$, where $C > 0$ is a constant:*

$$\int_\Omega u^2 \psi dx \leq 2K^2 \int_{B_R(x_0) \cap \Omega} u dx \int_\Omega u^{-1} |\nabla u|^2 \psi dx + K^2 \left(\frac{A^2}{2} + 1 \right) \|u\|_1^2 \tag{2}$$

$$\int_\Omega u^2 dx \leq \frac{2K^2}{\log s} \int_\Omega (u \log u + e^{-1}) dx \int_\Omega u^{-1} |\nabla u|^2 dx + 2K^2 \|u\|_1^2 + 3s^2 |\Omega| \tag{3}$$

$$\int_\Omega u^3\psi dx \leq \frac{72K^2}{\log s}\int_{B_R(x_0)\cap\Omega}(u\log u+e^{-1})dx\int_\Omega|\nabla u|^2\psi dx+C\|u\|^3_{L^1(B_R(x_0)\cap\Omega)}+10|\Omega|s^3. \tag{4}$$

The second equation of (CZ) is written as

$$-\Delta v + a(x,t)v = u \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

with $a(x,t) = 1+|v|^{p-1}$, and then we obtain $v \geq 0$ by $u \geq 0$ and the maximum principle. This implies

$$\|v\|_1 + \|v\|_p^p = \|u\|_1$$

and therefore, it holds that

$$-\Delta v + v = h, \quad h = u - |v|^{p-1}v \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{5}$$

with

$$\|h\|_1 \leq \|u\|_1 + \|v\|_p^p \leq 2\|u\|_1.$$

Then, L^1 estimate of [2] gives $\sup_{0\leq t<T_{\max}}\|v\|_{W^{1,q}} \leq C_q\|u\|_1$ for $q \in [1,2)$ and hence

$$\sup_{0\leq t<T_{\max}}\|v\|_q \leq C'_q\|u\|_1 \qquad \text{for } q \in [1,\infty) \tag{6}$$

by Sobolev's embedding theorem. Here, we emphasize that $\lambda = \|u\|_1$ is invariant in t.

3 Characterization of the blowup point

Henceforth, we assume $T_{\max} < \infty$ and denote the blowup set by $\mathcal{B}$. Using (6), we can show the following lemma similarly to [11].

Lemma 3.1 *We obtain $x_0 \in \mathcal{B}$ if and only if*

$$\limsup_{t\uparrow T_{\max}}\int_{B_R(x_0)\cap\Omega} u\log u dx = \infty$$

for any $0 < R \ll 1$.

Proof. The 'if' part is clear, because $x_0 \notin \mathcal{B}$ implies

$$\limsup_{t\uparrow T_{\max}}\int_{B_R(x_0)\cap\Omega} u\log u dx < \infty \tag{7}$$

for some $0 < R \ll 1$ by the definition.

For the 'only if' part to prove, we assume (7) and show $x_0 \notin \mathcal{B}$. First, we obtain

$$\limsup_{t\uparrow T_{\max}} \int_\Omega (u\log u)\psi dx < \infty$$

for $\psi = \varphi^6_{x_0,R',R}$, where $0 < R' < R$. Then, multiplying the first equation of (CZ) by $u\psi$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi dx + \int_\Omega |\nabla u|^2\psi dx + \int_\Omega u\nabla u\cdot\nabla\psi dx = \int_\Omega u\psi\nabla v\cdot\nabla u dx + \int_\Omega u^2\nabla v\cdot\nabla\psi dx. \tag{8}$$

The first term of the right hand side of (8) is equal to

$$\int_\Omega u\psi\nabla u\cdot\nabla v dx = -\frac{1}{2}\int_\Omega u^2\Delta v\cdot\psi dx - \frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi dx.$$

Then, using the second equation of (CZ), we obtain

$$\begin{aligned}
&\int_\Omega u\psi\nabla u\cdot\nabla v dx \\
&= \frac{1}{2}\int_\Omega u^2 v\psi dx - \frac{1}{2}\int_\Omega u^2|v|^{p-1}v dx + \frac{1}{2}\int_\Omega u^3\psi dx - \frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi dx \\
&\le \frac{1}{2}\int_\Omega u^3\psi dx - \frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\psi dx \\
&= \frac{1}{2}\int_\Omega u^3\psi dx + \frac{1}{2}\int_\Omega v\nabla(u^2)\psi dx + \frac{1}{2}\int_\Omega u^2 v\Delta\psi dx.
\end{aligned}$$

Since the second term of right side of (8) is equal to

$$\int_\Omega u^2\nabla\psi\cdot\nabla v dx = -\int_\Omega v\nabla(u^2)\cdot\nabla\psi dx - \int_\Omega u^2 v\Delta\psi dx,$$

it holds that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi dx + \int_\Omega |\nabla u|^2\psi dx + \int_\Omega u\nabla u\cdot\nabla\psi dx \\
&\quad = \frac{1}{2}\int_\Omega u^3\psi dx - \frac{1}{2}\int_\Omega v\nabla(u^2)\nabla\psi dx - \int_\Omega u^2 v\Delta\psi dx.
\end{aligned} \tag{9}$$

By using Young's inequality and the estimate of ψ, we have

$$\frac{1}{2}\left|\int_\Omega u^2 v\Delta\psi dx\right| \le \frac{1}{3}\int_\Omega u^3\psi dx + \frac{B^3}{6}\|v\|_3^3, \tag{10}$$

$$\left|\int_\Omega u\nabla u\cdot\nabla\psi dx\right| \le \frac{1}{4}\int_\Omega |\nabla u|^2\psi dx + \frac{1}{3}\int_\Omega u^3\psi dx + \frac{4A^6}{3}|\Omega|, \tag{11}$$

and

$$\frac{1}{2}\left|\int_\Omega v\nabla u^2\cdot\nabla\psi dx\right| \leq \frac{1}{4}\int_\Omega |\nabla u|^2\psi dx + \frac{1}{3}\int_\Omega u^3\psi dx + \frac{A^6}{48}|\|v\|_6^6. \tag{12}$$

Inequalities (9)-(12) are summarized by

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi dx + \frac{1}{2}\int_\Omega |\nabla u|^2\psi dx \leq \frac{3}{2}\int_\Omega u^3\psi dx + C_1. \tag{13}$$

Using (4) with $s \ll 1$, we obtain

$$\frac{d}{dt}\int_\Omega u^2\psi dx + \frac{1}{2}\int_\Omega |\nabla u|^2\psi dx \leq 2C_1.$$

This implies

$$\sup_{0\leq t<T_{\max}}\int_\Omega u^2\psi dx < \infty. \tag{14}$$

Multiplying the first equation of (CZ) by $u^2\psi$ and integrating by parts, we have

$$\frac{d}{dt}\int_\Omega w^2\psi dx + \frac{5}{3}\int_\Omega |\nabla w|^2\psi dx \leq 4\int_\Omega w^3\psi dx + C_2 \tag{15}$$

for $w = u^{3/2}$ in a similar way. Here, it holds that

$$\sup_{0\leq t<T_{\max}}\int_{B_{R'}(x_0)\cap\Omega} w\log w dx < \infty \quad \text{and} \quad \sup_{0\leq t<T_{\max}}\|w\|_{L^1\left(B_{R'(x_0)\cap\Omega}\right)} < \infty$$

by (15) and therefore, we can repeat the above argument. Thus, with u, R, and $\psi = \varphi^6_{x_0,R',R}$, replaced by w, R', and $\psi = \varphi^6_{x_0,R'',R'}$, it follows that

$$\sup_{0\leq t<T_{\max}}\|w\|^{2/3}_{L^2(B_r(x_0)\cap\Omega)} = \sup_{0\leq t<T_{\max}}\|u\|_{L^3(B_r(x_0)\cap\Omega)} < \infty$$

similarly to (14), where $R'' \in (0, R')$ and $r \in (0, R)$. This implies

$$\sup_{t\in[0,T_{\max})}\|h\|_{L^3(B_r(x_0)\cap\Omega)} < \infty$$

in (5) by (6), and then

$$\sup_{0\leq t<T_{\max}}\|v\|_{W^{2,3}(B_{r'}(x_0)\cap\Omega)} < \infty$$

follows for $r' \in (0, r)$ from the elliptic regulality. Since $R' \in (0, R)$ and $r' \in (0, R')$ are arbitrary, this guarantees

$$\sup_{0\leq t<T_{\max}}\|v\|_{C^1(B_r(x_0)\cap\Omega)} < \infty \tag{16}$$

for any $r \in (0, R)$ by Morrey's theorem. Repeating the argument once more, we have also

$$\sup_{0 \le t < T_{\max}} \|u\|_{L^4(B_r(x_0)\cap\Omega)} < \infty. \tag{17}$$

From this stage we use only the first equation of (CZ), and the proof is the same as that of [11]. For reader's convenience, we describe the story in short. First, taking $r' \in (0, r)$, $\psi = \varphi^6_{x_0,r',r}$, and $q \ge 1$, we multiply the first equation of (CZ) by $u^q\psi^{q+1}$:

$$\frac{1}{q+1}\frac{d}{dt}\int_\Omega (u\psi)^{q+1}dx = -\int_\Omega \nabla(u^q\psi^{q+1})\cdot\nabla u dx + \int_\Omega u\nabla(u^q\psi^{q+1})\cdot\nabla v dx = -I + II.$$

Then, it follows that

$$\begin{aligned} I &= \int_\Omega \left(qu^{q-1}\psi^{q+1}\nabla u + u^q\nabla\psi^{q+1}\right)\cdot\nabla u dx \\ &\ge \frac{2}{q+1}\int_\Omega \left|\nabla u_1^{\frac{q+1}{2}}\right|^2 dx - \frac{A^2(q+1)}{2}\|u_0\|_1^{1/3}\left(\int_\Omega u_1^{1+\frac{2}{3}q}\right)^{2/3}, \end{aligned}$$

where $u_1 = u\psi$. Next, using (16),

$$L \equiv \sup_{0\le t<T_{\max}} \|\nabla v\|_{L^\infty(B_r(x_0)\cap\Omega)} < \infty,$$

we obtain

$$II \le \frac{1}{q+1}\int_\Omega \left|\nabla u_1^{\frac{q+1}{2}}\right|^2 + 4L^2(q+1)\int_\Omega u_1^{q+1}dx + LA(q+1)\|u_0\|_1^{1/6}\left(\int_\Omega u_1^{1+\frac{6}{5}q}dx\right)^{5/6}$$

and hence

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_1^{q+1}dx &\le -\int_\Omega \left|\nabla u_1^{\frac{q+1}{2}}\right|^2 dx + C_3(q+1)^2\int_\Omega u_1^{q+1}dx \\ &+ C_3(q+1)^2\left\{\left(\int_\Omega u_1^{1+\frac{3}{2}q}\right)^{2/3} + \left(\int_\Omega u_1^{1+\frac{6}{5}q}\right)^{5/6}\right\}, \end{aligned}$$

Here, $C_3 > 0$ is independent of $q \ge 1$ and we apply Moser's iteration scheme [1]. Then, it follows that

$$\sup_{0\le t<T_{\max}} \|u_1\|_\infty \le C_4 \max\left\{\left(\sup_{0\le t<T_{\max}} \|u_1\|_4^4 + 1\right)^{1/4}, \|u_0\|_\infty + 1\right\}$$

and hence

$$\sup_{0\le t<T_{\max}} \|u_1\|_\infty = \sup_{0\le t<T_{\max}} \|u\psi_1\|_\infty < \infty$$

by (17). This implies $\limsup_{t\uparrow T_{\max}} \|u\|_{L^\infty(B_{r'}(x_0)\cap\Omega)} < \infty$ and therefore, $x_0 \notin \mathcal{B}$.

4 Proof of Theorem 1

The proof of Lemma 3.1 is valid even to $\varphi = 1$, and in this case we can show that

$$\limsup_{t\uparrow T} \int_\Omega u \log u dx < \infty \tag{18}$$

implies

$$\limsup_{t\uparrow T} \|u\|_\infty < \infty. \tag{19}$$

If (19) is the case, then the elliptic and parabolic regularity guarantees that the solution u is continued after $t = T$. Here, we shall show that (18) follows from

$$\liminf_{t\uparrow T} \int_\Omega u \log u dx < \infty \tag{20}$$

and consequently, $T_{\max} < \infty$ implies

$$\liminf_{t\uparrow T} \int_\Omega u \log u dx = \infty,$$

and hence $\lim_{t\uparrow T} \|u\|_\infty = \infty$ as is desired.

In fact, multiplying $\log u$ by the first equation of (CZ), we have

$$\frac{d}{dt}\int_\Omega u \log u dx + \int_\Omega u^{-1}|\nabla u|^2 + uv + uv^p dx = \int_\Omega u^2 dx. \tag{21}$$

The right-hand side is estimated from above by (3), and then it follows that

$$\frac{d}{dt}\int_\Omega u \log u dx + \left\{1 - \frac{2K^2}{\log s}\int_\Omega (u \log u + e^{-1}) dx\right\} \int_\Omega u^{-1}|\nabla u|^2 dx \leq C\|u_0\|_1^2 + 3s^2|\Omega|.$$

Taking $s = s(t) = \exp\left(2K^2 \int_\Omega u \log u + e^{-1})\right) dx > 1$, we obtain

$$\frac{dJ}{dt} \leq C\|u_0\|_1^2 + 3|\Omega| \exp\left(4K^2 J\right),$$

where $J = \int_\Omega (u \log u + e^{-1}) dx$. This inequality guarantees that (20) implies (18).

5 Proof of Theorem 2

Given $x_0 \in \overline{\Omega}$, we take $0 < R' < R <\ll 1$ and set $\psi = \varphi^6_{x_0,R',R}$. Let $G = G(x, x')$ be the Green's function defined by

$$(-\Delta_{x'} + 1)G = \delta(x' - x) \ \ (x' \in \Omega), \qquad \frac{\partial}{\partial \nu_{x'}} G = 0 \ \ (x' \in \partial\Omega)$$

for $x \in \Omega$. Using $G(x,x') = G(x',x)$, we can show the following [11].

Lemma 5.1 *It holds that* $\rho \in L^\infty(\Omega \times \Omega)$*, where*

$$\rho(x,x') = \nabla\psi(x) \cdot \nabla_x G(x,x') + \nabla\psi(x')\nabla_{x'} G(x,x').$$

Multiplying $u\psi$ to the first equation of (CZ), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (u\log u)\psi dx + \int_\Omega (u^{-1}|\nabla u|^2 + uv + uv^p)\psi dx \\ &\quad = \int_\Omega u^2\psi dx + \int_\Omega (-(\log u + 1)\nabla u + u\log u \nabla v)\cdot \nabla\psi dx \end{aligned}$$

similarly to (21). The right-hand side is treated by the method of [11]. Using (6), we obtain the following.

Lemma 5.2 *It holds that*

$$\frac{d}{dt}\int_\Omega (u\log u)\psi dx + \frac{1}{4}\int_\Omega u^{-1}|\nabla u|^2\psi dx \le 2\int_\Omega u^2\psi dx + C_5. \tag{22}$$

Then, the argument in the previous section can be localized. More precisely, we obtain

$$\frac{d}{dt}\int_\Omega (u\log u)\psi dx + \frac{1}{4}\left(1 - 16K^2\int_{B_R(x_0)\cap\Omega} udx\right)\int_\Omega u^{-1}|\nabla u|^2\psi dx \le C_6$$

by (22) and (2), and therefore, the assumption

$$\limsup_{t\uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} udx < \varepsilon_0 \equiv \frac{1}{16K^2}$$

implies

$$\limsup_{t\uparrow T_{\max}} \int_{B_{R'}(x_0)\cap\Omega} u\log udx \le \limsup_{t\uparrow T_{\max}} \int_\Omega (u\log u)\psi dx < \infty.$$

Then, it follows that $x_0 \notin \mathcal{B}$ from Lemma 3.1. In other words, each $x_0 \in \mathcal{B}$ admits the estimate

$$\limsup_{t\uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} udx \ge \varepsilon_0 \tag{23}$$

for any $0 < R \ll 1$.

Now, we shall replace this condition by

$$\liminf_{t\uparrow T_{\max}} \int_{B_R(x_0)\cap\Omega} udx \ge \varepsilon_0, \tag{24}$$

where $x_0 \in \mathcal{B}$ and $0 < R \ll 1$ are arbitrary. If this is the case,

$$\sharp\mathcal{B} \leq \frac{\|u_0\|_1}{\varepsilon_0} < \infty$$

because L^1 norm of u is invariant: $\|u(\cdot,t)\|_1 = \|u_0\|_1$, and then the proof is complete.

For this purpose, we use the first equation of (CZ) and derive

$$\frac{d}{dt}\int_\Omega u\psi dx = \int_\Omega u\Delta\psi dx + \int_\Omega u\nabla v \cdot \nabla\psi dx, \tag{25}$$

where $\left|\int_\Omega u\Delta\psi dx\right| \leq B\|u_0\|_{L^1}$ is obvious. Then, the second term of the right-hand side of (25) is equal to

$$\int_\Omega\int_\Omega u(x,t)\nabla\psi(x)\cdot\nabla_x G(x,x')\left[u(x',t) - v^p(x',t)\right]dx'dx$$

$$= \frac{1}{2}\int_\Omega\int_\Omega \rho(x,x')u(x,t)u(x',t)dxdx' - \int_\Omega\int_\Omega u(x,t)\nabla\psi(x)\cdot\nabla_x G(x,x')v^p(x',t)dx'dx.$$

Here, Lemma 5.1 implies

$$\left|\int_\Omega\int_\Omega \rho(x,y)u(x,t)u(y,t)dxdy\right| \leq \|\rho\|_\infty\|u_0\|_1^2,$$

while

$$\begin{aligned}\left|\int_\Omega\int_\Omega u(x,t)\nabla\psi(x)\cdot\nabla_x G(x,x')v^p(x',t)dx'dx\right| &\leq A\left\|u_0\right\|_1\left\|(-\Delta+1)^{-1}v^p\right\|_{W^{1,\infty}} \\ &\leq AC\left\|u_0\right\|_1^{p+1}\end{aligned}$$

holds by (6). Thus, we obtain

$$\left|\frac{d}{dt}\int_\Omega u\psi dx\right| \leq B\|u_0\|_{L^1} + \frac{1}{2}\|\rho\|_{L^\infty(\Omega\times\Omega)}\|u_0\|_1^2 + AC\left\|u_0\right\|_1^{p+1}$$

and hence the convergence

$$\lim_{t\uparrow T_{\max}}\int_\Omega u\psi dx = \int_\Omega u_0(x)\psi dx + \int_0^{T_{\max}}\left(\frac{d}{dt}\int_\Omega u(\cdot,t)\psi dx\right)dt.$$

Since $0 < R \ll 1$ is arbitrary in (23), we obtain (24).

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