

On Block Sequences of Banach Frames

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Abstract

Block sequences with respect to Banach frames have been defined and studied. Examples have been given to show that a block sequence with respect to a Banach frame need not always be a Banach frame. A sufficient condition under which a block sequence to a Banach frame is a Banach frame, has been given. Further, it has been proved that there always exists a quotient space having a Banach frame. Further, we prove that if E and F are isomorphic to each other, then E has a Banach frame if and only if F has a Banach frame. Finally, a sufficient condition under which an exact retro Banach frame is an unconditional basis has been given.

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1 Introduction

Duffin and Schaeffer [7] introduced frames for Hilbert spaces while addressing some problems in the theory of non-harmonic Fourier-series. Later, Daubechies,

Grossman and Meyer [6] found a new fundamental application to wavelets and Gabor transforms in which frames played an important role.

During the last two decades, besides existing applications such as signal processing, image processing etc. some new application to the theory of frames have been developed recently. Casazza [3] and Benedetto and Fickus [2] have studied frames in finite dimensional spaces which attracted more attention due to their use in signal processing. For results on signal reconstruction without phase information, one may refer to [1].

Coifman and Weiss [5] introduced the notion of atomic decomposition for function spaces. Feichtinger and Gröchenig [9] extended the notion of atomic decomposition to Banach spaces. Gröchenig [10] introduced a more general concept for Banach spaces called Banach frame. Banach frames were further studied in [4, 13].

In the present paper, we define block sequences with respect to Banach frames and observe with the help of examples that a block sequence with respect to a Banach frame need not be a Banach frame. Also, we give a sufficient condition under which the block sequence to a Banach frame is a Banach frame. Further, it has been proved that the quotient space $E/[f_n]_{\perp}$, where E is a Banach space and $\{f_n\} \subset E^*$, always have a Banach frame. Moreover, we prove that if E and F are Banach spaces isomorphic to each other, then E has a Banach frame if and only if F has a Banach frame. Finally, we give a sufficient condition under which an exact retro Banach frame for E^* is an unconditional basis for E .

2 Preliminaries

Throughout this paper E will denote a Banach space over the scalar field \mathbb{K} (\mathbb{R} or \mathbb{C}), E^* the conjugate space of E , $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E , $[\tilde{f}_n]$ the closed linear span of $\{f_n\}$ in the $\sigma(E^*, E)$ -topology, E_d and $(E^*)_d$, respectively, the associated Banach spaces of the scalar-valued sequences indexed by \mathbb{N} , and $\gamma_E(V)$ the greatest number r such that the unit ball $\{f \in V : \|f\| \leq 1\}$ of V is $\sigma(E^*, E)$ -dense in the ball $\{f \in E^* : \|f\| \leq r\}$ of E^* .

A sequence $\{x_n\}$ in E is said to be complete if $[x_n] = E$ and a sequence $\{f_n\}$ in E^* is said to be total over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

Definition 2.1 ([10]). Let E be a Banach space and E_d be an associated Banach space of scalar-valued sequences, indexed by \mathbb{N} . Let $\{f_n\} \subset E^*$ and $S : E_d \rightarrow E$ be given. The pair $(\{f_n\}, S)$ is called a Banach frame for E with respect to E_d if

- (i) $\{f_n(x)\} \in E_d$ for each $x \in E$,

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E \quad (2.1)$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in E.$$

The positive constants A and B , respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \rightarrow E$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.1) is called the *frame inequality*.

The Banach frame $(\{f_n\}, S)$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset E^*$ no longer a Banach frame for E , then $(\{f_n\}, S)$ is called an *exact Banach frame*.

The following result which is referred in this paper is listed in the form of a lemma.

Lemma 2.2 ([15]). *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E , then E is linearly isometric to the associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$, where the norm is given by $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$.*

Definition 2.3 ([11]). Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar valued sequences associated with E^* indexed by \mathbb{N} . Let $\{x_n\} \subset E$ and $T : (E^*)_d \rightarrow E^*$ be given. The pair $(\{x_n\}, T)$ is called a *Retro Banach frame for E^** with respect to $(E^*)_d$ if

- (i) $\{f(x_n)\} \in (E^*)_d$ for each $f \in E^*$,
- (ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \quad f \in E^* \quad (2.2)$$

(iii) T is a bounded linear operator such that $T(\{f(x_n)\}) = f$, $f \in E^*$.

The positive constants A and B , respectively, are called *lower* and *upper frame bounds* of the retro Banach frame $(\{x_n\}, T)$. The operator $T : (E^*)_d \rightarrow E^*$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.2) is called the *retro frame inequality*.

3 Block Sequences

Definition 3.1. Let $(\{f_n\}, S)$ ($\{f_n\} \subset E^*$, $S : E_d \rightarrow E$) be a Banach frame for E with respect to E_d and let $\{D_i\}$ be a sequence of finite subsets of \mathbb{N}

such that $D_i \cap D_j = \phi$, for all $i, j \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} D_i = \mathbb{N}$. Then the sequence $\{g_n\}$ defined by

$$g_n = \sum_{i \in D_n} \alpha_i f_i \neq 0, \quad n \in \mathbb{N}$$

where α_i , $i \in D_n$, $n \in \mathbb{N}$ are any scalars, is called a *Block sequence* with respect to the Banach frame $(\{f_n\}, S)$.

One may observe that a block sequence with respect to a Banach frame is a Banach Bessel sequence. However, it may or may not be a Banach frame.

Example 3.2. Let $E = \ell^\infty$ and $\{f_n\} \subset E^*$ be the sequence defined by

$$f_{2n}(x) = f_{2n-1}(x) = \xi_n, \quad \text{for all } n \in \mathbb{N}, x = \{\xi_n\} \in E.$$

Then, by Lemma 2.2, there exists an associated Banach space $E_d = \{\{f_n(x)\} : x \in E\}$ with norm $\|\{f_n(x)\}\|_{E_d} = \|x\|_E$, $x \in E$. Define $S : E_d \rightarrow E$ by $S(\{f_n(x)\}) = x$, $x \in E$. Then $(\{f_n\}, S)$ is a Banach frame for E with respect to E_d . Let $D_i = \{2i-1, 2i\}$, $i \in \mathbb{N}$. Then $D_i \cap D_j = \phi$, for all $i, j \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} D_i = \mathbb{N}$. Let $g_n = \sum_{i \in D_n} \alpha_i f_i$, $n \in \mathbb{N}$, where $\alpha_i = i$, $i \in D_n$, $n \in \mathbb{N}$. Then $\{g_n\}$ is a Block sequence with respect to the Banach frame $(\{f_n\}, S)$. Also, by Lemma 2.2, there exists an associated Banach space $E_{d_0} = \{\{g_n(x)\} : x \in E\}$ with norm $\|x\|_E = \|\{g_n(x)\}\|_{E_{d_0}}$, $x \in E$. Define $S_0 : E_{d_0} \rightarrow E$ by $S_0(\{g_n(x)\}) = x$, $x \in E$. Then $(\{g_n\}, S_0)$ is a Banach frame for E with respect to E_{d_0} .

Example 3.3. Let $E = \ell^\infty$ and let $\{f_n\} \subset E^*$ be a sequence defined by

$$f_n(x) = \begin{cases} \frac{1}{n} \xi_n, & \text{if } n \text{ is odd} \\ -\frac{1}{n} \xi_n, & \text{if } n \text{ is even, } n \in \mathbb{N}; x = \{x_n\} \subset E. \end{cases}$$

Then as in Example 3.2, by Lemma 2.2, there exists a reconstruction operator $S : E_d = \{\{f_n(x)\} : x \in E\} \rightarrow E$ given by $S(\{f_n(x)\}) = x$, $x \in E$ such that $(\{f_n\}, S)$ is a Banach frame for E . Let $\{D_i\}$, $\{g_i\}$ be as in Example 3.2 and $\alpha_i = i$, for all $i \in D_n$, $n \in \mathbb{N}$. Then, there exists no reconstruction operator $S_0 : E_d \rightarrow E$ such that $(\{g_n\}, S_0)$ is a Banach frame for E . Infact, there exists no associated Banach space E_{d_0} such that $(\{g_n\}, S_1)$ ($S_1 : E_{d_0} \rightarrow E$) is a Banach frame for E with respect to E_{d_0} .

In view of Examples 3.2 and 3.3, it is natural to ask for sufficient conditions under which a block sequence with respect to a Banach frame is a Banach frame. We prove the following result.

Theorem 3.4. *Let $(\{f_n\}, S)$ ($\{f_n\} \subset E^*, S : E_d \rightarrow E$) be an exact Banach frame for E and $\{g_n\}$ be a block sequence with respect to $(\{f_n\}, S)$ and let $L : E_d \rightarrow E_d$ is a bounded linear operator such that $L(\{f_n(x)\}) = \{g_n(x)\}$, $x \in E$. Then there exists a reconstruction operator $T : E_d \rightarrow E$ such that $(\{g_n\}, T)$ is an exact Banach frame for E if and only if there exists a positive constant λ such that*

$$\|\{g_n(x)\}\|_{E_d} \geq \lambda \|\{f_n(x)\}\|_{E_d}, \quad x \in E.$$

Proof. In view of Theorem 5.1 in [14], there exists a reconstruction operator $T : E_d \rightarrow E$ such that $(\{g_n\}, T)$ is a Banach frame for E with respect to E_d .

Since $(\{f_n\}, S)$ is exact, by Lemma 4.1 in [12], there exists a sequence $\{x_n\} \subset E$ such that $f_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Note that $f_n \neq 0$, $n \in \mathbb{N}$. Now

$$g_n = \sum_{i \in D_n} \alpha_i f_i \neq 0, \quad n \in \mathbb{N}.$$

Therefore, for each n , there exists an $\alpha_n \neq 0$. Define $\{y_n\} \subset E$ by $y_n = \frac{1}{\alpha_n} x_n$, $n \in \mathbb{N}$. Then $g_i(y_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Therefore, by Lemma 4.1 in [12] again, $(\{g_n\}, T)$ is an exact Banach frame for E . \square

Let $(\{f_n\}, S)$ ($\{f_n\} \subset E^*, S : E_d \rightarrow E$) be a Banach frame for E with respect to E_d . Let $\{g_n\} \subset E^*$ be given by

$$g_n = \sum_{i=1}^n \alpha_i f_i, \quad n \in \mathbb{N},$$

where α_i , $i \in \mathbb{N}$ be any scalars. Then there may not exists a reconstruction operator T such that $(\{g_n\}, T)$ ($T : E_{d_0} \rightarrow E$) is a Banach frame for E . Indeed, let $E = c_0$ and $\{f_n\} \subset E^*$ be defined by

$$f_n(x) = \xi_n, \quad n \in \mathbb{N}; \quad x = \{\xi_n\} \in E.$$

Then, by Lemma 2.2, there exists an associated Banach space $E_{d_1} = \{\{f_n(x)\} : x \in E\}$ with $\|\{f_n(x)\}\|_{E_{d_1}} = \|x\|$, $x \in E$ and a reconstruction operator $U : E_{d_1} \rightarrow E$ such that $(\{f_n\}, U)$ is a Banach frame for E . Take $\alpha_1 = 0$ and $\alpha_n = 1$, $n \geq 2$. Then $x = (1, 0, 0, \dots) \in E$ is such that $g_n(x) = 0$, for all $n \in \mathbb{N}$.

However, if $\alpha_n \neq 0$, for all $n \in \mathbb{N}$, in the definition of g_n . Then the above discussion leads us to the following result.

Theorem 3.5. *Let $(\{f_n\}, S)$ ($\{f_n\} \subset E^*, S : E_d \rightarrow E$) be a Banach frame for E with respect to E_d . Let $\{g_n\} \subset E^*$ be given by $g_n = \sum_{i=1}^n \alpha_i f_i$, $n \in \mathbb{N}$, where $\alpha_n \neq 0$, for all $n \in \mathbb{N}$, be any scalars. Then there exists an associated Banach space E_{d_0} and a reconstruction operator $T_0 : E_{d_0} \rightarrow E$ such that $(\{g_n\}, T_0)$ is a Banach frame for E .*

Remark 3.6. *The converse part of Theorem 3.5 is also true. In fact, if $\left(\left\{\sum_{i=1}^n \alpha_i f_i\right\}, S\right)$ $\left(\left\{\sum_{i=1}^n \alpha_i f_i\right\} \subset E^*, S : E_d \rightarrow E\right)$, where $\alpha_i \neq 0$ for each $i \in \mathbb{N}$ is a Banach frame for E , then there exists an associated Banach space E_{d_1} and a reconstruction operator $S_1 : E_{d_1} \rightarrow E$ such that $(\{f_n\}, S_1)$ is a Banach frame for E .*

4 Banach Frames

As observed in [12], the quotient space ℓ^∞/c_0 does not have a Banach frame. However, in the following theorem, we prove that there always exists a quotient space having a Banach frame.

We begin this section with the following results.

Theorem 4.1. *Let E be a Banach space and $\{f_n\} \subset E^*$ be any sequence. Then the quotient space $E/[f_n]_\perp$ has a Banach frame.*

Proof. Note that $([f_n]_\perp)^\perp$ is linearly isometric to $(E/[f_n]_\perp)^*$. Let $\{\phi_n\} \subset (E/[f_n]_\perp)^*$ be the image of $\{f_n\}$ under this isometry and let $W : E \rightarrow E/[f_n]_\perp$ be the quotient map. Then $\varphi_n(W(x)) = f_n(x)$, for all $x \in E$, $n \in \mathbb{N}$. Therefore, by Lemma 2.2, there exists an associated Banach space $E_d = \{\{\varphi_n(W(x))\} : x \in E\}$ with norm given by $\|W(x)\|_{E/[f_n]_\perp} = \|\{\varphi_n(W(x))\}\|_{E_d}$, $x \in E$. Define $S : E_d \rightarrow E/[f_n]_\perp$ by $S(\{\varphi_n(W(x))\}) = W(x)$, $x \in E$. Then $(\{\phi_n\}, S)$ is a Banach frame for $E/[f_n]_\perp$ with respect to E_d . \square

Next, we obtain a sufficient condition for two Banach spaces E and F such that E has a Banach frame if and only if F has a Banach frame.

Theorem 4.2. *Let E and F be any two isomorphic Banach spaces. Then E has a Banach frame if and only if F has a Banach frame.*

Proof. Let $(\{f_n\}, S)$ be a Banach frame for E . Let U be an isomorphism of E onto F . Put $(U^{-1})^*(f_n) = g_n$, $n \in \mathbb{N}$. Then $\{g_n\}$ is a sequence in F^* . Let $g_n(y) = 0$, for all $n \in \mathbb{N}$. Then $f_n(U^{-1}(y)) = 0$, for all $n \in \mathbb{N}$. So, by frame inequality for the Banach frame $(\{f_n\}, S)$, $y = 0$. Therefore, by Lemma 2.2, there exists an associated Banach space $F_d = \{\{g_n(y)\} : y \in F\}$ with norm $\|y\|_F = \|\{g_n(y)\}\|_{F_d}$, $y \in F$ and a reconstruction operator $T : F_d \rightarrow F$ given by $T(\{g_n(y)\}) = y$, $y \in F$ such that $(\{g_n\}, T)$ is a Banach frame for F with respect to F_d .

The converse part follow similarly. \square

Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \rightarrow E^*)$ be an exact retro Banach frame for E^* . Then, by Lemma 2.4 in [11], there exists a sequence $\{f_n\} \subset E^*$ such that $f_i(x_j) = \delta_{ij}$ (Kronecker delta), for all $i, j \in \mathbb{N}$. The sequence

$\{f_n\} \subset E^*$ is called the admissible sequence to the retro Banach frame $(\{x_n\}, T)$.

Finally, we prove the following result.

Theorem 4.3. *Let $(\{x_n\}, T)$ ($\{x_n\} \subset E, T : (E^*)_d \rightarrow E^*$) be an exact retro Banach frame for E^* with admissible sequence $\{f_n\} \subset E^*$. Let $\{n_k\}$ be any increasing sequence of positive integers such that $\gamma_{[x_{n_k}]}([f_{n_k}]) > 0$. Then $\{x_n\}$ is an unconditional basis of E .*

Proof. Under the hypothesis, the canonical mapping V of E onto $E/[x_j]_{j \in \mathbb{N} \setminus \{n_k\}}$ is an isomorphism. Further, since $(\{x_n\}, T)$ is a retro Banach frame for E^* , $V([x_{n_k}]) = E/[x_j]_{j \in \mathbb{N} \setminus \{n_k\}}$. So $(V|_{[x_{n_k}]})^{-1}V$ is a projection of E onto $[x_{n_k}]$ along $[x_j]_{j \in \mathbb{N} \setminus \{n_k\}}$, $k \in \mathbb{N}$. Thus $E = [x_{n_k}] \oplus [x_j]_{j \in \mathbb{N} \setminus \{n_k\}}$. Take $\{n_k\} = \{1, 2, \dots, k\}$, $k \in \mathbb{N}$. Then, along $[x_i]_{i \in \mathbb{N} \setminus K}$ for every set $K \subset \mathbb{N}$, there exists a projection p_k of E onto $[x_i]_{i \in K}$. Now, for any finite set I of indices and any scalars $\alpha_i (i \in I)$, $\sum_{i \in I \cap K} \alpha_i x_i \in [x_i]_{i \in K}$ and $\sum_{i \in I \cap (\mathbb{N} \setminus K)} \alpha_i x_i \in [x_i]_{i \in \mathbb{N} \setminus K}$.

Therefore

$$p_k \left(\sum_{i \in I} \alpha_i x_i \right) = \sum_{i \in I \cap K} \alpha_i x_i.$$

Hence $\{x_n\}$ is an unconditional basis for E . \square

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