

# Convergence and Stability of Three-Step Iteration Schemes with Errors for Generalized Nonlinear Complementarity Problems

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## **Abstract**

In this paper, a new class of generalized nonlinear complementarity problems is introduced and studied, and an iterative algorithm, called the three-step iteration scheme with errors, is suggested. The existence and uniqueness of solution for the generalized nonlinear complementarity problem and the convergence, stability and weak stability of iterative sequence generated by the algorithm are given.

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## **1 Introduction and Preliminaries**

It is known that the variational inequality theory and complementarity theory have lots of applications in diverse fields of mathematical, regional, physical, and engineering sciences (see [1], [3], [7]-[12], [14], [15] and the references therein). The convergence, stability and weak stability of the Mann

and Ishikawa iterative procedures for several kinds of nonlinear mappings and equations were studied by several researchers ([2], [4]-[6], [13], [16]). Bokhoven [14] used first the change of variables technique to study a class of linear complementarity problems in  $\mathbb{R}^n$ . Later, Ahmad, Kazmi and Rehman [1] modified the change of variables technique to suggest some iterative methods for solving to the implicit complementarity problem in Hilbert spaces. Liu and Kang [7] discussed the convergence and stability of a perturbed three-step iterative algorithm for a class of completely generalized nonlinear quasi-variational inequalities.

Inspired and motivated by the research work in [1]-[16], in this paper, we introduce and study a new class of generalized nonlinear complementarity problems in Hilbert spaces. Using the change of variables technique, we prove that the generalized nonlinear complementarity problem and the fixed point problem are equivalent. Using this equivalence, we suggest and analysis a new unified and general algorithm, which is called the three-step iteration scheme with errors, for computing the approximate solution of the generalized nonlinear complementarity problem. Under certain conditions, we establish the existence and uniqueness of solution of the generalized nonlinear complementarity problem, and the convergence, stability and weak stability of iterative sequence generated by the algorithm. Our results are extensions and improvements of previously known results.

Let  $H$  be a Hilbert space on which the inner product and norm are denoted by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$  respectively. Let  $K$  be a nonempty closed convex cone of  $H$ ,  $K^* = \{y \in H : \langle y, x \rangle \geq 0, \forall x \in K\}$  denote the convex polar cone of  $K$  in  $H$  and  $P_K$  stand for the projection of  $H$  onto  $K$ . Assume that  $T, A : H \rightarrow H$  and  $N : H \times H \rightarrow H$  are mappings. We now consider the problem of finding  $u \in H$  such that

$$u \in K, N(Tu, Au) \in K^* \quad \text{and} \quad \langle N(Tu, Au), u \rangle = 0, \quad (1.1)$$

which is called the *generalized nonlinear complementarity problem*.

Related to the generalized nonlinear complementarity problem (1.1), we consider the generalized nonlinear variational inequality as follows:

$$\text{Find } u \in K \text{ such that } \langle N(Tu, Au), v - u \rangle \geq 0, \quad \forall v \in K. \quad (1.2)$$

It is worth mentioning that problem (1.1) can be written as

$$u \in K, \quad v = N(Tu, Au) \in K^* \quad \text{and} \quad \langle v, u \rangle = 0. \quad (1.3)$$

Let us recall the following concepts and results. For each  $u \in H$ , we define the absolute value of  $u$  as follows:

$$|u| = u^+ + u^-, \quad u^+ = \sup\{0, u\} \quad \text{and} \quad u^- = -\inf\{0, u\}.$$

It is known that for any arbitrary element  $u \in H$ , we get that  $u = u^+ + u^-$  and  $\langle u^+, u^- \rangle = 0$ . For all  $z \in H$ , we consider the following change of variables:

$$u = \frac{|z| + z}{2} = z^+ = P_K(z), \quad v = \frac{|z| - z}{\rho} = \frac{2z^{-1}}{\rho} = \frac{2}{\rho}(P_K(z) - z), \quad (1.4)$$

where  $\rho > 0$  is a constant.

**Lemma 1.1.** [3] *Let  $z$  be an arbitrary element in  $H$  and  $K$  be a closed convex set of  $H$ . Then  $u \in K$  satisfies the inequality  $\langle u - z, v - u \rangle \geq 0$  for all  $v \in K$  if and only if  $u = P_K(z)$ . Moreover,  $P_K$  is nonexpansive.*

**Lemma 1.2.** [15] *Let  $K$  be a closed convex cone of  $H$ . Then*

- (a)  $\langle P_K(x) - x, y \rangle \geq 0, \forall (x, y) \in H \times K;$
- (b)  $\langle P_K(x) - x, P_K(x) \rangle = 0, \forall x \in K.$

**Lemma 1.3.** [4] *Let  $\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}$  and  $\{c_n\}_{n \geq 0}$  be nonnegative sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_nb_n + c_n, \quad \forall n \geq 0,$$

where  $\{t_n\}_{n \geq 0} \subset [0, 1], \sum_{n=0}^\infty t_n = \infty, \lim_{n \rightarrow \infty} b_n = 0$  and  $\sum_{n=0}^\infty c_n < \infty$ . Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 1.4.** *Let  $K$  be a closed convex cone of  $H$ ,  $t$  and  $\rho$  be positive constants with  $t \leq 1$ . Then the following statements are equivalent each other:*

- (a) *the generalized nonlinear complementarity problem (1.1) has a solution  $u \in K$ ;*
- (b) *the generalized nonlinear variational inequality (1.2) has a solution  $u \in K$ ;*
- (c) *there exists an element  $u \in K$  such that*

$$u = P_K\left(u - \frac{1}{2}\rho N(Tu, Au)\right); \quad (1.5)$$

- (d) *the mapping  $G : H \rightarrow H$  defined by*

$$G(z) = (1 - t)z + t\left(z^+ - \frac{1}{2}\rho N(Tz^+, Az^+)\right), \quad z^+ = P_K(z), \quad \forall z \in H \quad (1.6)$$

has a fixed point  $z \in H$ . Moreover,  $u = P_K(z)$ .

*Proof.* We first assume that problem (1.1) has a solution  $u \in K$ . Since  $N(Tu, Au) \in K^*$ , it follows that

$$\begin{aligned} \langle N(Tu, Au), v - u \rangle &= \langle N(Tu, Au), v \rangle - \langle N(Tu, Au), u \rangle \\ &= \langle N(Tu, Au), v \rangle \geq 0, \quad \forall v \in K. \end{aligned}$$

That is, problem (1.2) has a solution  $u \in K$ .

Conversely, suppose that problem (1.2) has a solution  $u \in K$ . Then  $2u$  and  $0$  are in  $K$ . Taking  $v = 2u$  and  $v = 0$  in (1.2), respectively, we have

$$\langle N(Tu, Au), u \rangle \geq 0 \quad \text{and} \quad \langle N(Tu, Au), u \rangle \leq 0,$$

which imply that  $\langle N(Tu, Au), u \rangle = 0$ . It follows that

$$0 \leq \langle N(Tu, Au), v - u \rangle = \langle N(Tu, Au), v \rangle, \quad \forall v \in K,$$

that is,  $N(Tu, Au) \in K^*$ . Therefore, problem (1.1) has a solution  $u \in K$ .

It follows from Lemma 1.1 that  $u \in K$  is a solution of problem (1.2) if and only if

$$\begin{aligned} \left\langle u - \left( u - \frac{1}{2}\rho N(Tu, Au) \right), v - u \right\rangle &\geq 0, \quad \forall v \in K \\ \Leftrightarrow u &= P_K \left( u - \frac{1}{2}\rho N(Tu, Au) \right). \end{aligned}$$

We next assume that (c) holds. Put

$$z = u - \frac{1}{2}\rho N(Tu, Au) \tag{1.7}$$

In view of (1.4) and (1.5), we know that (1.7) can be written as:

$$z = z^+ - \frac{1}{2}\rho N(Tz^+, Az^+), \tag{1.8}$$

where  $z^+ = P_K(z)$ . Note that  $t \in (0, 1]$ . Hence (1.8) is equivalent to

$$z = (1 - t)z + t \left( z^+ - \frac{1}{2}\rho N(Tz^+, Az^+) \right) \in G(z).$$

Consequently, (d) holds.

We finally assume that (d) holds. Using (1.4), we conclude that the following statements are true:

$$\begin{aligned} z \in G(z) &\Leftrightarrow z = z^+ - \frac{1}{2}\rho N(Tz^+, Az^+) \\ &\Leftrightarrow z^{-1} = \frac{1}{2}\rho N(Tz^+, Az^+) \\ &\Leftrightarrow v = N(Tu, Au). \end{aligned} \tag{1.9}$$

By virtue of (1.4), (1.9) and (a) of Lemma 1.2, we infer that  $u = P_K(z) \in K$  and  $v = N(Tu, Au) \in K^*$ . On the other hand, (1.4), (1.9) and (b) of Lemma 1.2 ensure that

$$\langle N(Tu, Au), u \rangle = \langle v, u \rangle = \left\langle \frac{2}{\rho}(P_K(z) - z), P_K(z) \right\rangle = 0.$$

Therefore, (a) holds. This completes the proof.  $\square$

Invoking Lemma 1.4, we suggest the following three-step iteration scheme with errors for the generalized nonlinear complementarity problem (1.1):

**Algorithm 1.5.** Let  $K$  be a closed convex cone of  $H$  and  $T, A : H \rightarrow H$  and  $N : H \times H \rightarrow H$  be mappings. Given  $z_0 \in H$ , compute the sequence  $\{z_n\}_{n \geq 0}$  by the iterative schemes

$$\begin{aligned} x_n &= (1 - \gamma_n)z_n + \gamma_n \left( z_n^+ - \frac{\rho}{2} N(Tz_n^+, Az_n^+) \right) + s_n, \quad z_n^+ = P_K(z_n), \\ y_n &= (1 - \beta_n)x_n + \beta_n \left( x_n^+ - \frac{\rho}{2} N(Tx_n^+, Ax_n^+) \right) + q_n, \quad x_n^+ = P_K(x_n), \\ z_{n+1} &= (1 - \alpha_n)z_n + \alpha_n \left( y_n^+ - \frac{\rho}{2} N(Ty_n^+, Ay_n^+) \right) + p_n, \quad y_n^+ = P_K(y_n) \end{aligned} \tag{1.10}$$

for all  $n \geq 0$ , and  $\{s_n\}_{n \geq 0}$ ,  $\{q_n\}_{n \geq 0}$  and  $\{p_n\}_{n \geq 0}$  are the sequences of the elements of  $H$  introduced to take into account possible inexact computations, and the sequences  $\{\alpha_n\}_{n \geq 0}$ ,  $\{\beta_n\}_{n \geq 0}$  and  $\{\gamma_n\}_{n \geq 0}$  satisfy

$$0 \leq \alpha_n, \beta_n, \gamma_n \leq 1 \quad \text{for all } n \geq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n = \infty \tag{1.11}$$

**Definition 1.6.** Let  $A : H \rightarrow H$  be a mapping. A mapping  $N : H \times H \rightarrow H$  is said to be

- (1) *r-strongly monotone* with respect to  $A$  in the first argument if there exists a constant  $r > 0$  such that  $\langle N(Ax, z) - N(Ay, z), x - y \rangle \geq r \|x - y\|^2$  for all  $x, y, z \in H$ ;
- (2)  *$\alpha$ -Lipschitz continuous in the first argument* if there exists a constant  $\alpha > 0$  such that  $\|N(x, z) - N(y, z)\| \leq \alpha \|x - y\|$  for all  $x, y \in H$ .

It follows from (1) and (2) that  $\alpha \geq \gamma$ . Similarly, we can define the Lipschitz continuity of  $N$  in the second argument.

**Definition 1.7.** ([2], [16]) Let  $T : H \rightarrow H$  be a mapping and  $\{\alpha_n\}_{n \geq 0}$  be a sequence in  $[0, 1]$ . Assume that  $x_0 \in H$  and  $x_{n+1} = f(T, \alpha_n, x_n)$  define an iteration procedure which yields a sequence of points  $\{x_n\}_{n \geq 0} \subset H$ . Suppose that  $F(T) = \{x \in H : x = Tx\} \neq \emptyset$  and  $\{x_n\}_{n \geq 0}$  converges to some  $u \in F(T)$ . Let  $\{z_n\}_{n \geq 0}$  be an arbitrary sequence in  $H$  and  $\varepsilon_n = \|z_{n+1} - f(T, \alpha_n, z_n)\|$  for all  $n \geq 0$ . If  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  implies that  $\lim_{n \rightarrow \infty} z_n = u$ , then the iteration procedure defined by  $x_{n+1} = f(T, \alpha_n, x_n)$  is said to be *T-stable* or *stable with respect to T*. If  $\varepsilon_n = \varepsilon'_n + \varepsilon''_n$  with  $\sum_{n=0}^{\infty} \varepsilon'_n < \infty$  and  $\varepsilon''_n = l_n \alpha_n$  for all  $n \geq 0$  and  $\lim_{n \rightarrow \infty} l_n = 0$  implies that  $\lim_{n \rightarrow \infty} z_n = u$ , then the iteration procedure  $\{x_n\}_{n \geq 0}$  is said to be *weakly T-stable*.

Zhou, Chang and Cho [16] pointed out that  $T$ -stability implies weak  $T$ -stability. Harder and Hicks [2] proved how such a sequence  $\{z_n\}_{n \geq 0}$  could arise in practice and demonstrated the importance of investigating the stability of various iterative schemes for various classes of nonlinear mappings.

## 2 Convergence, Stability and Weak Stability

We now study the existence and uniqueness of solution of the generalized non-linear complementarity problem (1.1) and establish the convergence, stability and weak stability of iterative sequences of generated by Algorithm 1.5

**Theorem 2.1.** *Let  $K$  be a closed convex cone of  $H$  and  $T, A : H \rightarrow H$  be  $\alpha$ -Lipschitz continuous and  $\beta$ -Lipschitz continuous, respectively. Let  $N : H \times H \rightarrow H$  be  $\gamma$ -Lipschitz continuous in the first argument and  $\delta$ -Lipschitz continuous in the second argument, and  $h$ -strongly monotone with respect to  $T$  in the first argument. Assume that*

$$\theta = \frac{1}{2}\rho\delta\beta + \sqrt{1 - \rho h + \frac{1}{4}\rho^2\gamma^2\alpha^2}, \tag{2.1}$$

$$\lim_{n \rightarrow \infty} \beta_n \|s_n\| = \lim_{n \rightarrow \infty} \|q_n\| = 0 \tag{2.2}$$

and one of the conditions (2.3) and (2.4) below holds:

$$\sum_{n=0}^{\infty} \|p_n\| < \infty, \tag{2.3}$$

there exists a nonnegative sequence  $\{d_n\}_{n \geq 0}$  such that

$$\|p_n\| = d_n \alpha_n \text{ for all } n \geq 0 \text{ and } \lim_{n \rightarrow \infty} d_n = 0. \tag{2.4}$$

Let  $\{f_n\}_{n \geq 0}$  be an arbitrary sequence in  $H$  and define  $\{\varepsilon_n\}_{n \geq 0} \subset [0, +\infty)$  by

$$\begin{aligned} \varepsilon_n &= \left\| f_{n+1} - \left[ (1 - \alpha_n)f_n + \alpha_n \left( g_n^+ - \frac{1}{2}\rho N(Tg_n^+, Ag_n^+) \right) + p_n \right] \right\|, \\ g_n &= (1 - \beta_n)f_n + \beta_n \left( h_n^+ - \frac{1}{2}\rho N(Th_n^+, Ah_n^+) \right) + q_n, \\ h_n &= (1 - \gamma_n)f_n + \gamma_n \left( f_n^+ - \frac{1}{2}\rho N(Tf_n^+, Af_n^+) \right) + s_n, \\ g_n^+ &= P_K(g_n), \quad h_n^+ = P_K(h_n), \quad f_n^+ = P_K(f_n) \end{aligned} \tag{2.5}$$

for all  $n \geq 0$ . If there exists a positive constant  $\rho$  satisfying

$$\rho\delta\beta < 2 \tag{2.6}$$

and one of the following conditions

$$\delta\beta < h, \quad \rho < 4(h - \delta\beta)(\gamma^2\alpha^2 - \delta^2\beta^2)^{-1}, \tag{2.7}$$

$$\alpha\gamma < \delta\beta, \quad \rho > 4(\delta\beta - h)(\delta^2\beta^2 - \alpha^2\gamma^2)^{-1}, \tag{2.8}$$

then

- (a) the mapping  $G$  defined by (1.6) has a unique fixed point  $z \in H$ ;
- (b) the sequence  $\{z_n\}_{n \geq 0}$  generated by Algorithm 1.5 converges strongly to  $z$ ;
- (c) the sequence  $\{z_n\}_{n \geq 0}$  generated by Algorithm 1.5 is weakly  $G$ -stable.

*Proof.* We first prove that (a) holds. Let  $x, y$  be arbitrary elements in  $H$ . Since  $N$  is  $\gamma$ -Lipschitz continuous in the first argument and  $\delta$ -Lipschitz continuous in the second argument, and  $h$ -strongly monotone with respect to  $T$  in the first argument, we deduce immediately that

$$\begin{aligned} & \|x^+ - y^+ - \frac{1}{2}\rho(N(Tx^+, Ax^+) - N(Ty^+, Ax^+))\|^2 \\ &= \|x^+ - y^+\|^2 - \rho\langle N(Tx^+, Ax^+) - N(Ty^+, Ax^+), x^+ - y^+ \rangle \\ &\quad + \frac{1}{4}\rho^2\|N(Tx^+, Ax^+) - N(Ty^+, Ax^+)\|^2 \\ &\leq \left(1 - \rho h + \frac{1}{4}\rho^2\gamma^2\alpha^2\right)\|x^+ - y^+\|^2 \end{aligned} \tag{2.9}$$

and

$$\|N(y^+, Ax^+) - N(y^+, Ay^+)\| \leq \delta\beta\|x^+ - y^+\|. \tag{2.10}$$

Using (1.6), (2.1), (2.9), (2.10) and the nonexpansivity of  $P_K$ , we infer that

$$\begin{aligned} & \|G(x) - G(y)\| \\ &= \left\| (1-t)(x-y) + t\left[x^+ - y^+ - \frac{1}{2}\rho(N(Tx^+, Ax^+) - N(Ty^+, Ay^+))\right] \right\| \\ &\leq (1-t)\|x-y\| + t\left\|x^+ - y^+ - \frac{1}{2}\rho(N(Tx^+, Ax^+) - N(Ty^+, Ax^+))\right\| \\ &\quad + \frac{1}{2}t\rho\|N(Ty^+, Ax^+) - N(Ty^+, Ay^+)\| \\ &\leq (1-t)\|x-y\| + t\left(\sqrt{1 - \rho h + \frac{1}{4}\rho^2\gamma^2\alpha^2} + \frac{1}{2}\rho\delta\beta\right)\|x^+ - y^+\| \\ &= (1-t)\|x-y\| + t\theta\|P_K(x) - P_K(y)\| \\ &\leq (1-t(1-\theta))\|x-y\|. \end{aligned}$$

Notice that (2.1) and (2.6) mean that

$$\begin{aligned} \theta < 1 &\Leftrightarrow 1 - \rho h + \frac{1}{4}\rho^2\gamma^2\alpha^2 < 1 - \rho\delta\beta + \frac{1}{4}\rho^2\delta^2\beta^2 \\ &\Leftrightarrow \frac{1}{4}\rho^2(\gamma^2\alpha^2 - \delta^2\beta^2) < \rho(h - \delta\beta). \end{aligned} \tag{2.11}$$

Now we consider the following three cases:

Case 1. Suppose that  $\delta\beta < h$ . Note that  $\gamma\alpha \geq h$ . Then (2.11) implies that

$$\theta < 1 \Leftrightarrow \rho < 4(h - \delta\beta)(\gamma^2\alpha^2 - \delta^2\beta^2)^{-1}.$$

Case 2. Suppose that  $\delta\beta = h$ . Since  $\gamma\alpha \geq h$ , it follows from (2.11) that

$$\theta < 1 \Leftrightarrow 0 \leq \frac{1}{4}\rho^2(\gamma^2\alpha^2 - \delta^2\beta^2) < \rho(h - \delta\beta) = 0,$$

which is a contradiction.

Case 3. Suppose that  $\delta\beta > h$ . If  $\gamma\alpha \geq \delta\beta$ , then (2.11) means that

$$\theta < 1 \Leftrightarrow 0 \leq \frac{1}{4}\rho^2(\gamma^2\alpha^2 - \delta^2\beta^2) < \rho(h - \delta\beta) < 0,$$

which is impossible. Hence  $\gamma\alpha < \delta\beta$ . According to (2.11), we get that

$$\theta < 1 \Leftrightarrow \rho > 4(\delta\beta - h)(\delta^2\beta^2 - \gamma^2\alpha^2)^{-1}.$$

Thus  $\theta < 1$  is equivalent to (2.7) and (2.8). It follows from (2.7), (2.8) and  $t \in (0, 1]$  that  $1 - t(1 - \theta) < 1$ . Hence  $G$  has a unique fixed point  $z \in H$ .

We next prove that (b) holds. Since  $z$  is the unique fixed point of  $G$ , we have

$$\begin{aligned} z &= (1 - \gamma_n)z + \gamma_n\left(z^+ - \frac{\rho}{2}N(Tz^+, Az^+)\right), \\ &= (1 - \beta_n)z + \beta_n\left(z^+ - \frac{\rho}{2}N(Tz^+, Az^+)\right), \\ &= (1 - \alpha_n)z + \alpha_n\left(z^+ - \frac{\rho}{2}N(Tz^+, Az^+)\right), \end{aligned} \tag{2.12}$$

where  $z^+ = P_K(z)$ . From (1.10), (2.9), (2.10) and (2.12), we conclude that

$$\begin{aligned} &\|x_n - z\| \\ &= \left\| (1 - \gamma_n)(z_n - z) \right. \\ &\quad \left. + \gamma_n\left[ z_n^+ - z^+ - \frac{1}{2}\rho(N(Tz_n^+, Az_n^+) - N(Tz^+, Az^+)) \right] + s_n \right\| \\ &\leq (1 - \gamma_n)\|z_n - z\| \\ &\quad + \gamma_n\left\| z_n^+ - z^+ - \frac{1}{2}\rho(N(Tz_n^+, Az_n^+) - N(Tz^+, Az_n^+)) \right\| \\ &\quad + \frac{1}{2}\gamma_n\rho\|N(Tz^+, Az_n^+) - N(Tz^+, Az^+)\| + \|s_n\| \\ &\leq (1 - \gamma_n)\|z_n - z\| + \gamma_n\theta\|z_n^+ - z^+\| + \|s_n\| \end{aligned}$$

$$\begin{aligned} &\leq (1 - \gamma_n(1 - \theta))\|z_n - z\| + \|s_n\| \\ &\leq \|z_n - z\| + \|s_n\|. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|y_n - z\| &\leq (1 - \beta_n)\|z_n - z\| + \beta_n\theta\|x_n^+ - z^+\| + \|q_n\| \\ &\leq \|z_n - z\| + \beta_n\|s_n\| + \|q_n\| \end{aligned}$$

and

$$\begin{aligned} &\|z_{n+1} - z\| \\ &\leq (1 - \alpha_n)\|z_n - z\| + \alpha_n\theta\|y_n^+ - z^+\| + \|p_n\| \\ &\leq (1 - \alpha_n(1 - \theta))\|z_n - z\| + \alpha_n(\beta_n\|s_n\| + \|q_n\|) + \|p_n\|. \end{aligned} \tag{2.13}$$

Suppose that (2.3) holds. Set  $a_n = \|z_n - z\|$ ,  $b_n = (1 - \theta)^{-1}(\beta_n\|s_n\| + \|q_n\|)$ ,  $c_n = \|p_n\|$  and  $t_n = (1 - \theta)\alpha_n$  for all  $n \geq 0$ . It follows from (1.11), (2.2), (2.3) and (2.13) and Lemma 1.3 that  $\lim_{n \rightarrow \infty} z_n = z$ .

Suppose that (2.4) holds. Put  $a_n = \|z_n - z\|$ ,  $b_n = (1 - \theta)^{-1}(\beta_n\|s_n\| + \|q_n\| + d_n)$ ,  $c_n = 0$  and  $t_n = (1 - \theta)\alpha_n$  for all  $n \geq 0$ . According to (1.11), (2.2), (2.4) and (2.13) and Lemma 1.3, we conclude that  $\lim_{n \rightarrow \infty} z_n = z$ .

We finally prove that (c) holds. Suppose that  $\varepsilon_n = \varepsilon'_n + \varepsilon''_n$  with  $\sum_{n=0}^{\infty} \varepsilon'_n < \infty$  and  $\varepsilon''_n = l_n\alpha_n$  for all  $n \geq 0$  and  $\lim_{n \rightarrow \infty} l_n = 0$ . As in the proofs of (a) and (b), from (2.5) and (2.12) we infer that

$$\begin{aligned} &\left\| (1 - \alpha_n)f_n + \alpha_n \left( g_n^+ - \frac{1}{2}\rho N(Tg_n^+, Ag_n^+) \right) + p_n - z \right\| \\ &\leq (1 - \alpha_n)\|f_n - z\| \\ &+ \alpha_n \left\| g_n^+ - z^+ - \frac{1}{2}\rho(N(Tg_n^+, Ag_n^+) - N(Tz^+, Ag_n^+)) \right\| \\ &+ \frac{1}{2}\alpha_n\rho\|N(Tz^+, Ag_n^+) - N(Tz^+, Az^+)\| + \|p_n\| \\ &\leq (1 - \alpha_n)\|f_n - z\| + \alpha_n\theta\|g_n - z\| + \|p_n\| \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \|g_n - z\| &\leq (1 - \beta_n)\|f_n - z\| + \beta_n\theta\|h_n - z\| + \|q_n\|, \\ \|h_n - z\| &\leq (1 - \gamma_n)\|f_n - z\| + \gamma_n\theta\|f_n - z\| + \|s_n\|. \end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.14), we have

$$\left\| (1 - \alpha_n)f_n + \alpha_n \left( g_n^+ - \frac{1}{2}\rho N(Tg_n^+, Ag_n^+) \right) + p_n - z \right\|$$

$$\leq (1 - \alpha_n(1 - \theta))\|f_n - z\| + \alpha_n(\beta_n\|s_n\| + \|q_n\|) + \|p_n\|. \quad (2.16)$$

In view of (2.5) and (2.16), we obtain that

$$\begin{aligned} & \|f_{n+1} - z\| \\ & \leq \varepsilon_n + \left\| (1 - \alpha_n)f_n + \alpha_n \left( g_n^+ - \frac{1}{2}\rho N(Tg_n^+, Ag_n^+) \right) + p_n - z \right\| \\ & \leq (1 - \alpha_n(1 - \theta))\|f_n - z\| + \alpha_n(\beta_n\|s_n\| + \|q_n\|) + \|p_n\| + \varepsilon_n. \end{aligned} \quad (2.17)$$

Suppose that (2.3) holds. Let  $a_n = \|f_n - z\|$ ,  $t_n = (1 - \theta)\alpha_n$ ,  $b_n = (1 - \theta)^{-1}(\beta_n\|s_n\| + \|q_n\| + l_n)$ ,  $c_n = \|p_n\| + \varepsilon'_n$  for each  $n \geq 0$ . It follows from (1.11), (2.2), (2.3), (2.17) and Lemma 1.3 that  $\lim_{n \rightarrow \infty} f_n = z$ .

Suppose that (2.4) holds. Let  $a_n = \|f_n - z\|$ ,  $t_n = (1 - \theta)\alpha_n$ ,  $b_n = (1 - \theta)^{-1}(\beta_n\|s_n\| + \|q_n\| + l_n + \|p_n\|)$ ,  $c_n = \varepsilon'_n$  for each  $n \geq 0$ . Using (1.11), (2.2), (2.4), (2.17) and Lemma 1.3, we know that  $\lim_{n \rightarrow \infty} f_n = z$ . This completes the proof.  $\square$

**Remark 2.2.** Under the assumptions of Theorem 2.1, by Lemma 1.4 we know that the generalized nonlinear complementarity problem (1.1) has a unique solution  $u = P_K(z) = \lim_{n \rightarrow \infty} P_K(z_n)$ , where  $z$  is the unique fixed point of  $G$  and  $\{z_n\}_{n \geq 0}$  satisfies (1.10).

We next study the stability of the three-step iteration sequence generated by Algorithm 1.5.

**Theorem 2.3.** *Let  $K$ ,  $N$ ,  $T$ ,  $A$  and  $\theta$  be as in Theorem 2.1 and (2.2) and (2.5) hold. Suppose that*

$$\lim_{n \rightarrow \infty} \|p_n\| = 0 \quad (2.18)$$

and

$$\text{there exists a constant } s \in (0, 1) \text{ such that } \alpha_n \geq s \text{ for all } n \geq 0. \quad (2.19)$$

If there exists a constant  $\rho > 0$  satisfying (2.6) and one of (2.7) and (2.8), then

(a) the mapping  $G$  defined by (1.6) has a unique fixed point  $z \in H$  and the sequence  $\{z_n\}_{n \geq 0}$  generated by Algorithm 1.5 converges strongly to  $z$ ;

(b)  $\lim_{n \rightarrow \infty} f_n = z$  if and only if  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ .

*Proof.* Let  $d_n = \|p_n\|\alpha_n^{-1}$  for all  $n \geq 0$ . Then (2.18) and (2.19) yield that (2.4) holds. Thus the conclusion of (a) follows from (a) and (b) of Theorem 2.1. It is easy to verify that (2.16) and (2.17) hold.

Suppose that  $\lim_{n \rightarrow \infty} f_n = z$ . Then (2.2), (2.16), (2.18) and (2.19) ensure that

$$\begin{aligned} \varepsilon_n &\leq \|f_{n+1} - z\| + \left\| (1 - \alpha_n)f_n + \alpha_n \left( g_n^+ - \frac{1}{2} \rho N(Tg_n^+, Ag_n^+) \right) + p_n - z \right\| \\ &\leq \|f_{n+1} - z\| + (1 - \alpha_n(1 - \theta))\|f_n - z\| + \alpha_n(\beta_n\|s_n\| + \|q_n\|) + \|p_n\| \\ &\leq \|f_{n+1} - z\| + (1 - s(1 - \theta))\|f_n - z\| + \beta_n\|s_n\| + \|q_n\| + \|p_n\| \\ &\rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . That is,  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ .

Conversely, suppose that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . Then (2.2) and (2.17)-(2.19) imply that

$$\begin{aligned} &\|f_{n+1} - z\| \\ &\leq (1 - \alpha_n(1 - \theta))\|f_n - z\| + \alpha_n(\beta_n\|s_n\| + \|q_n\|) + \|p_n\| + \varepsilon_n \\ &\leq (1 - s(1 - \theta))\|f_n - z\| + \beta_n\|s_n\| + \|q_n\| + \|p_n\| + \varepsilon_n. \end{aligned} \quad (2.20)$$

Put  $a_n = \|f_n - z\|$ ,  $b_n = s^{-1}(1 - \theta)^{-1}(\beta_n\|s_n\| + \|q_n\| + \|p_n\| + \varepsilon_n)$ ,  $c_n = 0$  and  $t_n = s(1 - \theta)$  for all  $n \geq 0$ . It follows from (2.2), (2.18), (2.20) and Lemma 1.3 that  $\lim_{n \rightarrow \infty} f_n = z$ . This completes the proof.  $\square$

**Remark 2.4.** Theorem 2.3 reveals that under suitable conditions the three-step iterative sequence generated by Algorithm 1.5 is  $G$ -stable.

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